Generalized continued fractions: Definitions, Convergence and Applications to Markov Chains

Habilitationsschrift

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Contents

Chapter 1

Introduction – Markov chains and continued fractions

Markov chains are used as mathematical models in various areas of applications. Among these are

- queueing systems (production networks, telecommunication, ...)
- epidemiology,
- biochemical stochastic reaction networks,
- ...

In practical applications, we are interested in computing long-run characteristics of Markov chains, for example the long-run average number of customers in a queueing system. Unfortunately, in most situations we are not able to obtain explicit representations of these characteristics, and thus, we have to use numerical procedures. In most realistic and thus detailed models, the state space of the Markov chain is very large. Often, there are infinitely many states. In these situations, the application of numerical methods becomes difficult.

If the transition probability matrix or generator matrix of the Markov chain has a block-tridiagonal structure, literature [BT95] suggests using matrix-analytic solution techniques for computing the invariant distribution (from which we can obtain long-run characteristics), and it is well-established that these methods are strongly related to matrix-valued continued fractions [Han99]. Similarly, for band-structured matrices, techniques relying on appropriate generalizations of (real-valued) continued fractions were introduced [Han92].

The block-tridiagonal or band structure of a transition probability or generator matrix can be interpreted as a restriction of the dynamic behaviour of the Markov chain: a transition from state i to state j can only occur if state j is in some kind of neighbourhood of state i. In this thesis, we will drop this restriction: We will introduce an appropriate definition of *generalized continued fractions* (gcfs) which enables us to represent long-run characteristics of Markov chains with arbitrary transition structures in terms of gcfs. We will discuss

- practical issues, that is, benefits of these representations, and

- theoretical issues, that is, we will compare our definition of gcfs (which is motivated by the application to Markov chains) with generalizations of continued fractions found in the literature, and we will derive convergence criteria and speed-of-convergence estimates for gcfs which are independent of the application to Markov chains.

In chapter 2, we will discuss the relationship between continued fractions and Markov chains with tridiagonal transition structures in a detailed manner, and we will introduce our definition of gcfs.

Before we will start with these technical details, we will briefly present the evolution of continued fractions throughout the last centuries (see section 1.2), and we will demonstrate which problems can arise when computing long-run characteristics of Markov chains with large state spaces by means of numerical procedures.

Note that in the applications which we have mentioned above, we usually use continuous-time Markov chains as mathematical models. Concerning the computation of long-run characteristics, we will see that the same problems arise for discrete-time and continuous-time Markov chains, and the same methods for solving these problems apply, see chapter 5 and chapter 6. Therefore, in this introductory chapter, we will focus on Markov chains in discrete time. For details on some terms which we will use in the next sections (irreducibility, recurrence, positive recurrence, ...), we refer to appendix C, in particular to section C.2.

1.1 Markov chains: Invariant measures and hitting times

The dynamics of a discrete-time Markov chain $(X_m)_{m\in\mathbb{N}_0}$ with discrete state space E is characterized by the matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$ of the one-step transition probabilities $p_{ij} = \mathbb{P}(X_{m+1} = j | X_m = i)$. Many long-run characteristics of the process can be written in terms of

- invariant measures,
- hitting probabilities and
- mean hitting times.

1.1.1 Invariant measures

If the Markov chain is irreducible and recurrent, the system $\psi\mathbf{P} = \psi$ of linear equations has a solution which is unique up to constant multiples, and which can be chosen strictly positive. Any vector $\psi > 0$ with this property is said to be an *invariant measure*. If additionally, $\sum_{i\in E} \psi_i = 1$, we refer to ψ as *invariant distribution* or *stationary distribution*. Such an invariant distribution exists in case of positive recurrence.

Under the conditions of irreducibility and recurrence, the *Ergodic Theorem* holds (we refer to appendix C for more details and more Limit Theorems for Markov chains): If ψ is any invariant measure, and $f, g : E \to \mathbb{R}$ are functions such that the sums $\psi f := \sum_{i\in E} \psi_i f(i)$ and

ψg converge absolutely with $\psi g \neq 0$, we have

$$\lim_{m\to\infty} \frac{\sum_{k=0}^{m-1} f(X_k)}{\sum_{k=0}^{m-1} g(X_k)} = \frac{\psi f}{\psi g}.$$

almost surely. In particular, in case of positive recurrence, let π denote the invariant distribution. If πf converges, we have

$$\lim_{n\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} f(X_k) = \pi f = \frac{\psi f}{\psi \mathbf{1}}$$

almost surely where $\mathbf{1}$ denotes the constant function with value 1.

As an example, let X_n denote the number of customers in a discrete-time queueing system at time n. In the long run, we may be interested in

- the average proportion of time in which the server is idle, that is

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \mathbf{1}_{\{0\}}(X_k) = \pi \mathbf{1}_{\{0\}} = \pi_0 = \frac{\psi \mathbf{1}_{\{0\}}}{\psi \mathbf{1}},$$

  where $\mathbf{1}_A$ is the indicator function of set A.

- the average number of customers in the system, that is

$$\lim_{n\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} X_k = \pi \mathrm{id} = \frac{\psi \mathrm{id}}{\psi \mathbf{1}},$$

  where id is the identity.

These considerations motivate that in applications, it is highly important to be able to compute an invariant measure ψ or the value of ψf for some function f. Unfortunately, in most applications, there is no explicit representation of ψ or ψf, and thus, we have to use numerical methods.

For applying these methods to Markov chains with (infinitely) large state spaces, we have to truncate the state space in an appropriate way. Without restriction, we assume $E = \mathbb{N}_0$. Then it is reasonable to choose some N, and use the matrix $(p_{ij})_{i,j=0}^{N}$ for computing an approximation for ψ, say $\psi^{(N)}$. Finally, we approximate

$$\psi f \approx \psi^{(N)} f := \sum_{n=0}^{N} \psi_n^{(N)} f(n).$$

A further problem which arises in the context of computing invariant measures numerically, is *instability*. In order to demonstrate this effect, we consider a Markov chain with state space $E = \mathbb{N}_0$ and tridiagonal transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} & & & \\ p_{10} & p_{11} & p_{12} & & \\ & p_{21} & p_{22} & p_{23} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let ψ be an invariant measure with $\psi_0 = 1$. Then $\psi\mathbf{P} = \psi$ can be rewritten as

$$\psi_0 p_{00} + \psi_1 p_{10} = \psi_0, \tag{1.1.1}$$

$$\psi_{n-1} p_{n-1,n} + \psi_n p_{nn} + \psi_{n+1} p_{n+1,n} = \psi_n, \qquad n \in \mathbb{N}. \tag{1.1.2}$$

Due to $p_{00} + p_{01} = 1$ and $p_{n,n-1} + p_{nn} + p_{n,n+1} = 1$ for $n \geq 1$, we obtain

$$\psi_n = \prod_{k=1}^{n} \frac{p_{k-1,k}}{p_{k,k-1}}, \qquad n \in \mathbb{N}_0$$

with an easy induction. For a moment, assume that we did not know this explicit representation of the solution. Then we would try to compute ψ numerically, and the easiest way to do so is 'forward computation', that is, we rewrite (1.1.1) and (1.1.2) as $\psi_1 = \psi_0 \cdot \frac{1-p_{00}}{p_{10}}$ and

$$\psi_{n+1} = \frac{\psi_n(1 - p_{nn}) - \psi_{n-1} p_{n-1,n}}{p_{n+1,n}}, \qquad n = 1, 2, 3, \ldots$$

and compute $\psi_1, \psi_2, \psi_3, \ldots$ using this recurrence scheme.

For $p_{n,n+1} = \frac{1}{2 \cdot 9^n}$ and $p_{n,n-1} = \frac{10}{2 \cdot 9^n}$, that is,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{10}{2 \cdot 9} & 1 - \frac{11}{2 \cdot 9} & \frac{1}{2 \cdot 9} & & & \\ & \frac{10}{2 \cdot 9^2} & 1 - \frac{11}{2 \cdot 9^2} & \frac{1}{2 \cdot 9^2} & & \\ & & \frac{10}{2 \cdot 9^3} & 1 - \frac{11}{2 \cdot 9^3} & \frac{10}{2 \cdot 9^3} & \\ & & & \ddots & & \ddots \end{pmatrix},$$

the invariant measure ψ with $\psi_0 = 1$ is given by $\psi_n = \left(\frac{9}{10}\right)^n$, but the general solution of the second-order recurrence scheme is

$$\psi_n = c_1 \cdot 9^n + c_2 \left(\frac{9}{10}\right)^n$$

with constants c_1, c_2: Note that

$$\psi_{n+1} = \frac{\psi_n \cdot \frac{11}{2 \cdot 9^n} - \psi_{n-1} \frac{1}{2 \cdot 9^{n-1}}}{\frac{10}{2 \cdot 9^{n+1}}}$$

simplifies to the second-order linear difference equation

$$\psi_{n+1} - \psi_n \cdot \frac{99}{10} + \psi_n \cdot \frac{81}{10} = 0$$

with constant coefficients. The corresponding characteristic polynomial has the roots $\frac{9}{10}$ and 9.

Theoretically, the initial condition $\psi_0 \cdot \frac{1}{2} + \psi_1 \cdot \frac{10}{2 \cdot 9} = \psi_0$ guarantees $\psi_1 = \psi_0 \cdot \frac{9}{10}$, and thus, $c_1 = 0$. Unfortunately, slight numerical errors result in the forward computation becoming more and more influenced by the increasing solution 9^n.

n	exact solution ψ_n	forward computation
0	1	1
1	0.9	0.9
2	0.81	0.81
3	0.729	0.729
4	0.6561	0.6561
5	0.5905	0.5905
6	0.5314	0.5314
7	0.4783	0.4783
8	0.4305	0.4305
10	0.3487	0.3487
12	0.2824	0.2824
15	0.2059	0.2066
16	0.1853	0.1918
17	0.1668	0.2250
18	0.1501	0.6738
19	0.1351	4.848
20	0.1216	42.54
25	0.07179	$2.505 \cdot 10^6$
30	0.04239	$1.479 \cdot 10^{11}$
40	0.01478	$5.157 \cdot 10^{20}$
50	0.005154	$1.798 \cdot 10^{30}$

The computation is performed using data type `double` in `C++`. It is equivalent to the computation of the invariant measure for the continuous-time Markov chain in Example 6.3.2 in section 6.3, see remark at the end of the discussion in Example 6.3.2.

From a practical point of view, two questions arise:

- Is there a way to overcome the numerical instability?
- How to choose N? Is there any chance to obtain results on the error $\left|\psi f - \psi^{(N)} f\right|$?

Literature suggests a relationship between numerically stable methods for obtaining invariant measures for transition probability matrices with block-tridiagonal or band structure, and appropriate generalizations of continued fractions, see [Han99; Han92] and the discussion in section 5.4.

Hence, an approach for dealing with arbitrarily structured transition probability matrices is finding an appropriate generalization of continued fractions that allows us to derive numerically stable algorithms for this general setting. Finding this definition is an important issue in this thesis, we will present our approach in chapter 2. It will turn out that this definition allows us to represent ψf as a generalized continued fraction, and $\psi^{(N)} f$ as its Nth approximant. Therefore, the answer to the second question is given by speed-of-convergence statements for generalized continued fractions.

Actually, for $f \geq 0$ we will find some C_N such that we can guarantee

$$\psi^{(N)} f \leq \psi f \leq \psi^{(N)} f + C_N,$$

and in many cases, both lower and upper bound can be computed efficiently.

In section 1.2, we will reflect the development of continued fractions throughout the last centuries. In particular, we will emphasize on the relationship to subdominant solutions of second-order difference equations. In section 2.1, we will present more details on this relationship in the context of the computation of invariant measures. We will use these considerations to obtain a generalization of continued fractions which is capable of dealing with transition probability matrices which do not necessarily have a tridiagonal structure.

Before we will proceed with these topics, we will briefly demonstrate that the other interesting long-run characteristics, that is, hitting probabilities, mean hitting times, ..., can also be characterized as a (minimal, and often subdominant) solution of a system of linear equations.

1.1.2 Hitting probabilities

Let $0 \in E$, let $\tau_0 = \inf\{m \in \mathbb{N} : \ X_m = 0\}$ be the first *hitting time* on state 0, let $\beta_0 = 1$ and $\beta_n = \mathbb{P}(\tau_0 < \infty | X_0 = n)$ for $n \in E \setminus \{0\}$. Intuitively, $(\beta_n)_{n\in\mathbb{N}_0}$ fulfills the system

$$x_k = \sum_{n\in E} p_{kn} x_n, \qquad k \neq 0, \tag{1.1.3}$$

and in fact, this conjecture is true, see Theorem C.2.1 (or [MT93, Proposition 8.4.1]). For infinitely many states, there may be more than one solution of this system of equations with $x_0 = 1$. Then, according to Theorem C.2.1, we can characterize $(\beta_n)_{n\in\mathbb{N}_0}$ as the **minimal positive** solution of (1.1.3) subject to $x_0 = 1$.

As an easy example, consider a gambler with initial wealth $n \in \mathbb{N}$ and an opponent with infinite wealth (e.g. a bank). The gambler wins each bet (and wins a coin) with probability p and loses each bet (and a coin) with probability $1 - p$. The game will end if the gambler has no money left. If X_m denotes the gambler's wealth after m bets, $(X_m)_{m\in\mathbb{N}_0}$ is a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & & & \\ 1-p & 0 & p & & \\ & 1-p & 0 & p & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then (1.1.3) simplifies to the second-order difference equation

$$x_k = p x_{k+1} + (1-p) x_{k-1}, \qquad k \geq 1.$$

The general solution is given by

$$x_k = \begin{cases} \gamma_1 + \gamma_2 \left(\frac{1-p}{p}\right)^k, & p \neq \frac{1}{2}, \\ \gamma_1 + \gamma_2 k, & p = \frac{1}{2} \end{cases}$$

with some constants $\gamma_1, \gamma_2 \in \mathbb{R}$. Since $(\beta_n)_{n\in\mathbb{N}_0}$ is the minimal positive solution with $\beta_0 = 1$, we obtain

$$\beta_k = \begin{cases} 1, & p \leq \frac{1}{2}, \\ \left(\frac{1-p}{p}\right)^k, & p > \frac{1}{2}. \end{cases}$$

In any case, (β_k) is dominated by some other solution, that is, $\lim_{k\to\infty} \frac{\beta_k}{x_k} = 0$ for

$$x_k = \begin{cases} \left(\frac{1-p}{p}\right)^k, & p < \frac{1}{2}, \\ k, & p = \frac{1}{2}, \\ 1, & p > \frac{1}{2}. \end{cases}$$

Due to the solution being dominated by another solution of the difference equation, the 'naive' forward computation will become unstable again.

Nevertheless, in more complicated situations, we have to use numerical methods for solving (1.1.3) since there will be no explicit formula for the hitting probabilities. For example, in epidemiological models, the probability of a disease to die out eventually is a hitting probability, but the transition structure of the underlying Markov models is much more complicated than in the gambling example.

For infinitely many states, we have to truncate the system of equations again, and the same questions concerning numerical instability and error bounds as for invariant measures arise. Actually, there is a direct analog to computing ψf: In many applications, we will be interested in the total probability of ever reaching the (absorbing) state 0 for a given initial distribution $\alpha = (\alpha_n)_{n\in E}$ with $\alpha_n = \mathbb{P}(X_0 = n)$. This total probability is

$$\mathbb{P}(\tau_0 < \infty \text{ or } X_0 = 0) = \alpha_0 + \sum_{n\neq 0} \alpha_n \mathbb{P}(\tau_0 < \infty | X_0 = n) = \sum_{n\in E} \alpha_n \beta_n = \alpha\beta.$$

We will see that in many situations, we will find efficient ways to compute an approximation for $\alpha\beta$ or lower and upper bounds, see chapter 6.

Remark 1.1.1. We have focused on probabilities for ever reaching state 0. Naturally, by renaming, hitting probabilities for any state $\in E$ can be obtained. Furthermore, if we are interested in the probability of reaching a subset $C \subset E$ eventually, this probability does not depend on the dynamics within C. Hence, we may replace the set C by a single state 0.

1.1.3 Mean hitting times

In situations where we know that we will reach state 0 eventually, the distribution of the hitting time τ_0 becomes interesting. In particular, let $\mu_n = \mathbb{E}[\tau_0 | X_0 = n]$ for $n \neq 0$. Then intuitive considerations suggest that $(\mu_k)_{k\in\mathbb{N}}$ satisfies the system of equations

$$x_k = 1 + \sum_{n\neq 0} p_{kn} x_n, \qquad k \neq 0,$$

and again, this conjecture is true, and $(\mu_k)_{k\in\mathbb{N}}$ is the minimal positive solution, see Theorem C.2.3 for a more general result. The statement in Theorem C.2.3 allows a slight generalization: Instead of the time until hitting state 0 for the first time, we can consider costs measured by a function f which arise before reaching state 0, that is, we consider

$$\mu_n = \mathbb{E}\left[\sum_{m=0}^{\tau_0 - 1} f(X_m) | X_0 = n\right],$$

and obtain the system

$$\mu_k = f(k) + \sum_{n\neq 0} p_{kn}\mu_n, \qquad k \neq 0. \tag{1.1.4}$$

If α denotes the initial distribution of the Markov chain, we want to compute

$$\mathbb{E}\left[\sum_{m=0}^{\tau_0-1} f(X_m)|X_0 \neq 0\right] = \sum_{n\neq 0} \alpha_n\mu_n = \alpha\mu.$$

Again, we have to ask similar questions, that is, how to deal with instability and infinite state spaces.

1.2 Notes on the history of continued fractions

In this section, we want to give a brief introduction into the history of continued fractions and their generalizations. Concerning the evolution of continued fraction before 1900, we follow the discussions in [Per54; Per57; Ras91].

Simple continued fractions

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots \cfrac{}{b_{N-1} + \cfrac{1}{b_N}}}} \qquad \text{or} \qquad b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}}$$

with $b_0 \in \mathbb{Z}$, $b_1, b_2, \ldots \in \mathbb{N}$ were introduced in order to approximate physical constants. First systematic approaches are due to John Wallis (1616–1703) and Christiaan Huygens (1629–1695). Leonhard Euler (1707–1783) studied continued fractions deeply and proved

Theorem 1.2.1. *Finite simple continued fractions represent rational numbers, and for any rational number, there are exactly two representations as a finite simple continued fraction. On the other hand, infinite simple continued fractions converge to irrational numbers, and for any irrational number, there is a unique representation as an infinite simple continued fraction.*

In principle, the proof is based on Euklid's algorithm, and in fact, the coefficients b_n of the continued fraction representation of a real number can be obtained from applying Euklid's algorithm. Therefore, this result is often referred to as *Euler-Euklid-theorem*.

If $b_n = b$ for all $n \geq n_0$ for an infinite simple continued fraction, it can be written as

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots \cfrac{}{b_{N-1} + \cfrac{1}{X}}}},$$

where $X = b_N + \frac{1}{X}$, that is, X satisfies the quadratic equation $z^2 - b_N z - 1 = 0$. It can be shown that the continued fraction itself satisfies a quadratic equation with integer coefficients, that is, it is a quadratic irrational number. Precisely, Joseph–Louis de Lagrange (1736–1813) proved

Theorem 1.2.2. *Any simple periodic continued fraction is a quadratic irrational number, and any quadratic irrational number has a representation as a simple periodic continued fraction.*

Furthermore, Lagrange used that the approximants

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{\ddots \; b_2 + \cfrac{1}{b_{N-1} + \cfrac{1}{b_N}}}}$$

for infinite simple continued fractions can be written as $\frac{A_N}{B_N}$, where A_n and B_n both satisfy the recurrence relation $X_n = b_n X_{n-1} + X_{n-2}$. He proved that the fractions $\frac{A_N}{B_N}$ yield the sequence of best rational approximations for the irrational number which is represented by the corresponding infinite continued fraction.

In the 19th century, mathematicians generalized the concept of continued fractions, and introduced structures

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ldots}},$$

where $a_n, b_n \in \mathbb{C}$. In contrast to simple continued fractions, convergence is not guaranteed, and hence criteria had to be found, we state some of them here, for details we refer to [Per54; Per57].

Theorem 1.2.3 (Worpitzky, 1865)**.** *Let $b_n = 1$ for all $n \in \mathbb{N}$, and let $|a_n| \leq \frac{1}{4}$ for all $n \in \mathbb{N}$. Then the continued fraction converges.*

Theorem 1.2.4 (Pringsheim-Śleszyński, end of 19th century)**.** *Let $|b_n| \geq |a_n| + 1$ for all $n \in \mathbb{N}$. Then the continued fraction converges.*

Theorem 1.2.5 (Pincherle)**.** *Let there be two solutions $(Y_n), (Z_n)$ for $X_n = b_n X_{n-1} + a_n X_{n-2}$ with $\frac{Y_n}{Z_n} \to 0$ as $n \to \infty$. Then the continued fraction converges.*

The proofs are based on the representation $\frac{A_N}{B_N}$ for the approximants, where A_n and B_n both satisfy $X_n = b_n X_{n-1} + a_n X_{n-2}$. Amongst others, a reason for considering continued fractions with complex elements is the relationship to second-order difference equations of the form $x_n = b_n x_{n+1} + a_{n+1} x_{n+2}$: The equation can be written as

$$\frac{x_n}{x_{n+1}} = b_n + \frac{a_{n+1}}{\frac{x_{n+1}}{x_{n+2}}},$$

and hence, it is a natural idea to claim that

$$\frac{x_0}{x_1} = b_0 + \frac{a_1}{\frac{x_1}{x_2}} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} = K$$

For example, this approach is discussed in [Per09; Per57; Cas80; Gau67; Gau72]. In fact, it turns out that the continued fraction K converges if and only if the difference equation has a

subdominant solution $(y_n)_{n\in\mathbb{N}_0}$, that is, there is another solution $(z_n)_{n\in\mathbb{N}_0}$ with $\lim_{n\to\infty}\frac{y_n}{z_n}=0$. In this situation, we have $K=\frac{y_0}{y_1}$. In other words, the continued fraction characterizes an initial condition which is fulfilled by a subdominant solution. By shifting the index, we obtain

$$\frac{y_n}{y_{n+1}} = b_n + \cfrac{a_{n+1}}{b_{n+1}+\cfrac{a_{n+2}}{b_{n+2}+\ddots}} = K_n.$$

for all $n\in\mathbb{N}_0$. From an algorithmic point of view, we obtain a numerically stable algorithm for computing the subdominant solution $(y_n)_{n\in\mathbb{N}_0}$: Compute approximations for K_n, and use these approximations in the first-order recurrence scheme $y_{n+1}=\frac{y_n}{K_n}$.

An alternative for obtaining subdominant solutions of second-order difference equations is Miller's backward computation algorithm for linear difference equations, see [Mil68; Zah77], and indeed, both approaches are strongly related (see section 8.1).

Note that the relationship to difference equations directly applies in the context of Markov chains: If we want to compute invariant measures or absorption probabilities (for state 0) for a Markov chain with tridiagonal transition probability matrix $\mathbf{P}=(p_{ij})_{i,j\in\mathbb{N}_0}$, we have to find a minimal solution of the second-order difference equation

$$\begin{aligned}\psi_n &= \psi_{n-1}p_{n-1,n}+\psi_n p_{nn}+\psi_n p_{n,n+1}, \qquad n\geq 1 \text{ or}\\ x_n &= p_{n,n-1}x_{n-1}+p_{nn}x_n+p_{n,n+1}x_{n+1}, \qquad n\geq 1, \text{ respectively.}\end{aligned}$$

In section 2.1, we will come back to this relationship.

In the 1960s and 1970s (see [Wyn63; Wyn64; Pfl66; Fai71; Fai72]), continued fraction in non-commutative structures began to become interesting. For non-commutative continued fractions of the (one-sided) form

$$b_0 + a_1\left(b_1+a_2(b_2+\ldots)^{-1}\right)^{-1},$$

convergence results can be found in [Hay74; Neg76; RK00; SI99; ZZ03], and the two-sided case

$$b_0 + a_1\left(b_1+a_2(b_2+\ldots)^{-1}c_2\right)^{-1}c_1$$

was studied in [DR82; Sch96]. Amongst others, a reason for the evolution of continued fraction in non-commutative algebras was the desire to solve difference equations in these algebras, for example, obtaining continued-fraction based solutions of second-order matrix-vector difference equations. Such systems arise in the context of computing invariant measures or hitting probabilities for Markov chains with block-tridiagonal transition probability matrices (see section 1.3), another application can be found in [Bob07].

Another type of generalizing continued fractions is based on the representation $\frac{A_N}{B_N}$ for the approximants. Numerator and denominator are subject to the same linear second-order recurrence scheme, and allowing higher-order recurrence schemes will yield generalizations of continued fractions. The first approach in this direction is due to Carl Gustav Jacob Jacobi (1804–1851). He intended to characterize cubic irrational numbers by appropriate generalizations of simple continued fractions, see [Jac68]. Oskar Perron (1880–1975) generalized his idea, and obtained 'Jacobi chains', which he used for characterizing solutions of difference equations of order >2. This method is known as *Jacobi-Perron algorithm*, and we refer

to [Ber71] for a discussion of the algorithm and its evolution. In [Per20], Perron proved that 'infinite Jacobi chains' are capable of solving *sum equations*, that is, infinite difference equations.

In a similar, but slightly different way, Marcel de Bruin introduced *generalized continued fractions* or n-fractions (see [Bru74; Bru78]). Pringsheim-type convergence criteria were given in [Lev86; Lev89], and in [Cru79; Cru82], the relationship to difference equations of order $n > 2$ was established.

In some way, in [LB96; Ahl96], the ideas of generalizing both the domain (matrix algebras or general Banach algebras) and the recursion scheme were combined. We will omit further details here. Instead, we refer to the literature reviews at the end of chapter 3 and chapter 7 where we will give more details.

Here, more or less, we follow Perron's approach, and we will combine the idea of his 'infinite Jacobi chains' with the generalization of the domain, that is, we consider generalized continued fractions in Banach algebras. We will see that our definition and our convergence criteria include

- the classic term of continued fraction,
- continued fractions in non-commutative algebras [Wyn63; Wyn64; Pfl66; Fai71; Fai72; DR82; Sch96],
- infinite Jacobi chains [Per20] and generalized continued fractions in de Bruin's sense [Bru74; Bru78; Lev86; Lev89; Cru79; Cru82]
- and the structures considered in [LB96; Ahl96]

as special cases.

Many of the ideas in this thesis are inspired by methods for computing stationary distributions for Markov chains: In [Han92], Thomas Hanschke considered band-structured transition probability matrices with upper bandwidth 1 for Markov chains, and he proved that the stationary distribution can be represented in terms of generalized continued fraction.

Later on, he described an algorithm for computing the stationary distribution of a Markov chain with block–tridiagonal transition probability matrix or generator matrix [Han99]. In this algorithm (which is similar to a method in [BT95], we will present the basic algorithm in section 2.1, and some more details in section 6.2), he explicitly used the relationship between non-generalized matrix-valued continued fractions and solutions for second-order (vector-matrix) difference equations.

It turned out that the properties of (block-)tridiagonal transition probability matrices for irreducible Markov chains guarantee convergence of the continued fractions which are algorithmically used. More than that, for block size 1, the (real-valued) continued fraction meets a Pringsheim–type criterion, that is, Pringsheim's and Śleszyński's results could be used for proving the convergence, and on the other hand, considerations concerning Markov chains could be used for finding new proofs for Pringsheim–type criteria. For block size > 1, the continued fractions do not meet a Pringsheim–type criterion anymore, but they still converge. In [Bau10; Bau15], the author used this relationship for stating a Pringsheim–type convergence criterion for (non-generalized) continued fractions in Banach algebras.

In this thesis, one aim will be proving Pringsheim–type (and Worpitzky–type and Pincherle–type) criteria for generalized continued fractions. Again, we will use ideas which arise in the context of Markov chains. Conversely, the speed-of-convergence statements we will present may be used for estimating errors in numerical algorithms for Markov chains.

As pointed out above, for simple continued fractions, $b_n = b$ for all $n \geq n_0$ yields quadratic irrational numbers. For arbitrary continued fractions in $\mathbb{C}$ (or any Banach algebra), *periodicity* is characterized by $b_n = b$ and $a_n = a$ for all $n \in \mathbb{N}$. In a similar way, periodicity for generalized continued fractions can be defined, and we will see that a special case is given by matrix–geometric methods for computing stationary distributions of level-independent quasi-birth-death processes. In the literature reviews at the end of chapter 5, we will refer to these methods more explicitly. Matrix–geometric methods were introduced by Neuts [Neu81], and some of our results can be interpreted as direct generalizations of Neuts considerations. In section 9.3, we will explicitly use this relationship.

1.3 Block matrices and Markov chains

Block-structured transition probability matrices occur in applications in which the state space E is partitioned into *levels*, that is $E = \dot{\bigcup}_{n\in\mathbb{N}_0} E_n$. If p_{mn} denotes the matrix of one-step transition probabilities from states in E_m to states in E_n, we obtain $\mathbf{P} = (p_{mn})_{m,n\in\mathbb{N}_0}$ as the transition probability matrix, and the p_{mn} are matrices of dimension $|E_m| \times |E_n|$.

Partitions of the state space often arise in a natural way. A typical example is given by Markov models for a queue with a Markovian arrival process as input and/or phase-type distributed service times, see [Neu81; BB05; BKF14] for more details. For some of these models, exact representations of the invariant measure exist (see [He14] for an explicit invariant distribution for the $M/PH/1$ model), some other models can be solved by means of Neuts matrix-geometric methods [Neu81]. In more complicated situations, numerical methods have to be applied.

In many of the situations where partitions are given in a natural way, the structure of the block-matrix is quite easy. For example, for basic queueing models with phase-type distributions and Markovian arrival processes, we obtain block-tridiagonal structures. For this reason, many methods have been developed for computing invariant measures for such special cases, for example [BT95; Han99]. As pointed out in section 1.2, in [Han99] the relationship to matrix-valued continued fractions was established. The reason for this relationship is as follows: If we set $\psi_n = (\psi_i)_{i\in E_n}$, we can still write

$$\psi_n = \sum_{n=0}^{\infty} \psi_m p_{mn},$$

and in case of a block-tridiagional transition probability matrix $\mathbf{P}$, we obtain the second-order difference equation

$$\psi_n = \psi_{n-1}p_{n-1,n} + \psi_n p_{nn} + \psi_{n+1}p_{n+1,n}$$

again. As mentioned in section 1.2, it is a natural attempt to represent a solution of this difference equation in terms of a continued fraction built up by the coefficients p_{mn}. Since

these coefficients are matrices, this approach results in a matrix-valued continued fraction. In section 2.1, we will give more details.

For new Markovian models, the availability of these methods makes it quite attractive to find such a partition of the state space that the corresponding block-matrix of transition probabilities (or transition rates for continuous-time Markov chains) has tridiagonal (or similar) structure. We briefly refer to two further examples based on continuous-time Markov chains.

- Consider a queueing system with finite capacity. Customers which are rejected due to lack of waiting capacity will join the 'orbit' of retrying customers and will eventually try to join the queue again. If R_t denotes the number of customers in the orbit and N_t denotes the number of customers in the queue or at service at time t, the process $((R_t, N_t))_{t\geq 0}$ is a continuous-Markov chain under some assumptions (exponentially distributed service times, interarrival times, ...), but transitions with positive rate will change the number of customers in the orbit at most by 1. If the capacity of the queue (including servers) is K, the state space $E = \mathbb{N}_0 \times \{0, \dots, K\}$ can be partitioned into $E = \bigcup E_n$ where $E_n = \{n\} \times \{0, \dots, K\}$. For further issues on this model and extensions, we refer to [Her08], and for an extensive discussion of retrial queues, we refer to [AGC08].

- Consider an epidemiological model where S_t is the number of sane population members (susceptibles) and I_t is the number of infectives at time t. The state space of the process $((S_t, I_t))_{t\geq 0}$ is $\mathbb{N}_0 \times \mathbb{N}_0$ (or some subset), and by defining $E_n = \{(i,j) \in \mathbb{N}_0^2 : \ i+j = n\}$ or $E_n = \{(i,j) : \ \max(i,j) = n\}$, we will obtain continuous-time Markov chains with block-tridiagonal generator matrix under some modelling assumptions. For a more detailed description of the model and extensions, we refer to [BS16] and references therein.

These examples give reason for considering block-structured transition probability or generator matrices, and hence, matrix-valued continued fractions. Often these considerations result in numerical schemes for computing invariant measures (or absorption probabilities, mean hitting times, ...), but in some cases, the representation of invariant measures (or absorption probabilities, ...) by means of continued fractions or their generalizations can be exploited theoretically.

If not using the relationship to continued fractions algorithmically, there is no need for the levels E_n to be finite. Actually, the levels E_n can be uncountably infinite. If p_{mn} shall still describe the transition probabilities from states in E_m to states in E_n, p_{mn} is a *kernel* of probabilities $p_{mn}(x, A)$ for steps from state $x \in E_m$ to sets $A \in \mathcal{E}_n$, where $\mathcal{E}_n$ is a σ-algebra on E_n. In section 5.1, we will give more details. Here, we only consider a simple example which we will reconsider in section 9.4 at the end of this thesis.

Example 1.3.1. Consider the $G/D/1$ queue with iid interarrival times $T_1, T_2, \dots$ with continuous distribution function F_T and deterministic service times $S_1 = S_2 = \dots = 1$, and let W_n be the waiting time of the nth customer. According to *Lindley's recursion*, we have

$$W_{n+1} = \max\{W_n + 1 - T_{n+1}, 0\}, \qquad n \in \mathbb{N}_0.$$

$(W_n)_{n\in\mathbb{N}_0}$ is a Markov process in discrete time with values in $E = [0, \infty)$. For $t \geq 0$, we have

$$\mathbb{P}(W_{n+1} \leq t | W_n = s) = \mathbb{P}(W_n + 1 - T_{n+1} \leq t | W_n = s) = \mathbb{P}(s + 1 - T_{n+1} \leq t) = 1 - F_T(s + 1 - t).$$

Now define $E_n = [n, n+1)$. Then we obtain a partition of the state space, and $p_{mn} = 0$ for $n \geq m+2$ since the waiting time will increase at most by 1 in each step (from customer n to customer $n+1$). Hence, the resulting matrix $\mathbf{P}$ has lower Hessenberg structure

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} & & & \\ p_{10} & p_{11} & p_{12} & & \\ p_{20} & p_{21} & p_{22} & p_{23} & \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}.$$

In total, there will be practical applications where $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$ is a matrix of scalars, a matrix of matrices or a matrix of kernels. In order to deal with all these situations, it makes sense to develop a theory which is valid in any situation where $p_{ij} \in \mathcal{R}$ with some Banach algebra $\mathcal{R}$. This is a major issue of this thesis.

1.4 Organization of the results

We organize our results as follows.

In chapter 2, we will give more details concerning the relationship between Markov chains with (block-)tridiagonal transition structure and continued fractions. Using this relationship as a motivation, we present our definition of generalized continued fractions (gcfs). Afterwards, in chapter 3, we will directly prove Pringsheim–type and (as a corollary) Worpitzky–type convergence criteria. Furthermore, we will obtain speed-of-convergence statements.

The definition of gcfs is motivated by computing invariant measures for Markov chains with infinitely many states, and invariant measures fulfill systems of linear equations. Therefore, it makes sense that in chapter 4, we will investigate in which situations gcfs will define solutions for infinite systems of linear equations, say $\mathbf{H}x = 0$. In special cases, we will see that the solution generated by gcfs turns out to be minimal.

In chapter 5, we will apply our method to Markov chains: Hitting probabilities and mean hitting times fulfill a system of the form $\mathbf{H}x = 0$, and hence, we will use results from chapter 4 for finding gcf-based representation of these values. Similarly, invariant measures fulfill systems of the form $\psi\mathbf{H} = 0$, and hence, we will use transposed results for deriving gcf-based representations for invariant measures ψ, and in particular, for ψf. The speed-of-convergence statements for gcfs will allow us to determine computable lower and upper bounds for ψf in case that $f \geq 0$.

The gcf-based representations of hitting probabilities, mean hitting times and invariant measures will be exploited algorithmically in chapter 6. In particular, we will see that we are able to compute lower and upper bounds for ψf silmutaneously and efficiently in special cases which often occur in practical applications.

In the second part, we will consider a subspecies of gcfs, and in chapter 7, we will prove that we can represent the approximants for these gcfs as a usual fraction $A_N B_N^{-1}$, where (A_n) and (B_n) both satisfy the same recurrence scheme. After giving a specialized proof of the Pringsheim–type convergence criterion, we will demonstrate how to establish a wide range of generalizations of continued fractions we found in the literature as special cases of our definition.

For the gcfs discussed in chapter 7, we will consider the relationship to infinite systems of linear equations again; this is done in chapter 8. Here, we will focus on the relationship between infinite systems of linear equations and the corresponding *adjoint* systems. In this context, we will prove Pincherle–type convergence criteria.

In chapter 9, we will discuss periodic gcfs which turn out to be related to roots of holomorphic (or meromorphic) functions. We conclude our considerations with an example from queueing theory.

Finally, in chapter 10, we will present some problems and possible directions of further research. Up to this last chapter 10, we will conclude each chapter with a literature review in which we will compare the results derived in this thesis with previous statements and methods.

Part I

A general approach

Chapter 2

Defining generalized continued fractions

In this chapter, we want to develop a definition of generalized continued fractions (gcfs). We have already mentioned that such definitions can be found in the literature, but there is no unique definition. Our construction will cover a wide range of these definitions (see section 7.4), and hence, we will refer to the definition presented in this chapter as gcf.

The key idea is as follows:

- In the introduction, we have mentioned that for tridiagonal transition probability matrices, invariant measures and hitting probabilities are characterized as a minimal solution of second-order difference equations, and thus, can be represented in terms of continued fractions. In section 2.1, we will give more details on this relationship.
- In particular, we will discuss a probabilistic interpretation (as a probability or an expectation) for the continued fractions which occur in this context.
- We will rewrite the probabilistic interpretation in terms of the entries p_{ij} of the transition probability matrix. This idea leads to the concept of S-series which we will define in section 2.2.
- In Theorem 2.2.1, we will derive a recursion scheme for these series in case of unconditional convergence. For tridiagonal $\mathbf{P}$, the scheme simplifies to a continued fraction $b_0 + a_1 \left(b_1 + \ldots\right)^{-1} c_1$.
- In section 2.3, we will use the recursion scheme for the S-series as definition of gcfs – regardless on whether or not the series converge. In order to obtain 'traditional' letters, we will replace $1 - p_{nn} = b_n$, $-p_{mn} = a_{mn}$ for $m < n$ and $p_{mn} = c_{mn}$ for $m > n$.
- In this definition, there is no more need for $p_{mn}, b_n, a_{mn}, c_{mn}$ to be real or complex numbers. Instead, we will consider elements from an arbitrary Banach algebra with unity.

Througout the chapter, let $\mathcal{R}$ be a Banach algebra with unity I and let $\mathcal{R}^* = \{r : \ r^{-1} \text{ exists}\}$ be the set of all invertible elements. Furthermore, we introduce the notation $\mathbb{N}_{n,N} := \{n, \ldots, N\}$ for $n, N \in \mathbb{N}_0$, $n \leq N$, and we define $\mathbb{N}_{n,N} = \emptyset$ for $n > N$.

2.1 Markov Chains and continued fractions

In this section, we want to demonstrate the relationship between algorithms for computing invariant measures of Markov chains with tridiagonal transition structure and continued fractions. In principle, the algorithm which we discuss in this section was stated in [BT95; Han99]. Here, we use slightly different notations, and we will emphasize on the question where continued fractions arise in this context.

Consider a block-tridiagonal stochastic matrix

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} & & & \\ p_{10} & p_{11} & p_{12} & & \\ & p_{21} & p_{22} & p_{23} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with blocks $p_{ij} \in \mathbb{R}^{d\times d}$, that is, $\mathbf{P}$ can be interpreted as the transition probability matrix of some two-dimensional Markov Chain $((X_n, U_n))_{n\geq 0}$ with state space $\mathbb{N}_0 \times \{1, \dots, d\}$. In case of irreducibility and recurrence, we are interested in computing an invariant measure $\psi = (\psi_n)_{n\in\mathbb{N}_0}$, where $\psi_n = ((\psi_{(n,u)}))_{u=1}^d$. The system $\psi\mathbf{P} = \psi$ can be rewritten as

$$\psi_0 p_{00} + \psi_1 p_{10} = \psi_0, \tag{2.1.1}$$

$$\psi_{n-1}p_{n-1,n} + \psi_n p_{nn} + \psi_{n+1}p_{n+1,n} = \psi_n, \qquad n \in \mathbb{N}. \tag{2.1.2}$$

Since for $d > 1$,

- ψ_0 is a vector, and we are only allowed to fix one component, say $\psi_{(0,1)} = 1$,
- and we have no guarantee for the matrices $p_{n+1,n}$ to be invertible,

there is no explicit representation of the invariant measure, and forward computation (which would be unstable anyway) is not possible. Nevertheless, in applications, we want to compute the invariant measure numerically.

Hence, we have to truncate the state space or, equivalently, the system $\psi\mathbf{P} = \psi$. The simplest approach is replacing the infinite matrix $\mathbf{P}$ by the finite matrix $(p_{ij})_{i,j=0}^N$. Since this matrix is not stochastic, it will not have the eigenvalue 1 anymore. For this reason, we define $\psi^{(N)} = \left(\psi_n^{(N)}\right)_{n=0}^N$ to be a solution of

$$\sum_{m=0}^{N} \psi_m^{(N)} p_{mn} = \psi_n^{(N)}, \qquad n = 1, \dots, N.$$

Note that we explicitly exclude $n = 0$. Since $\mathbf{P}$ is block-tridiagonal, we obtain (2.1.2) for $n = 1, \dots, N-1$, and

$$\psi_{N-1}^{(N)} p_{N-1,N} + \psi_N^{(N)} p_{NN} = \psi_N^{(N)}. \tag{2.1.3}$$

Equation (2.1.3) directly yields

$$\psi_N^{(N)} = \psi_{N-1}^{(N)} p_{N-1,N} \left(I - p_{NN}\right)^{-1},$$

provided that this inverse exists. Together with (2.1.2) for $n = N-1$, we derive

$$\psi_{N-2}^{(N)} p_{N-2,N-1} = \psi_{N-1}^{(N)}(I - p_{N-1,N-1}) - \psi_{N-1}^{(N)} p_{N-1,N} \left(I - p_{NN}\right)^{-1} p_{N,N-1},$$

that is,

$$\psi_{N-1}^{(N)} = \psi_{N-2}^{(N)} p_{N-2,N-1} \left(I - p_{N-1,N-1} - p_{N-1,N} \left(I - p_{NN}\right)^{-1} p_{N,N-1}\right)^{-1},$$

provided that this inverse exists, too. Recursively, we obtain

$$\psi_n^{(N)} = \psi_{n-1}^{(N)} p_{n-1,n} \left(K_n^{(N)}\right)^{-1}, \qquad 1 \le n \le N,$$

where

$$\begin{aligned} K_N^{(N)} &= I - p_{NN}, \\ K_n^{(N)} &= I - p_{nn} - p_{n,n+1} \left(K_{n+1}^{(N)}\right)^{-1} p_{n+1,n}, \qquad 0 \le n < N. \end{aligned}$$

In particular, if the limit $\lim\limits_{N\to\infty} K_0^{(N)}$ exists, it has the form

$$b_0 + a_1 \left(b_1 + a_2 \left(b_2 + \ldots\right)^{-1} c_2\right)^{-1} c_1,$$

that is, it is a two-sided continued fraction.

Once we have determined $\psi_0^{(N)}$, we can use $\psi_n^{(N)} = \psi_{n-1}^{(N)} p_{n-1,n} \left(K_n^{(N)}\right)^{-1}$ for obtaining $\psi_1^{(N)}, \ldots, \psi_0^{(N)}$. Hence, the remaining task is computing $\psi_0^{(N)}$. For this purpose, we use the equation (2.1.1) which was ignored up to now. We obtain

$$\psi_0^{(N)} p_{00} + \psi_0^{(N)} p_{01} \left(K_1^{(N)}\right)^{-1} p_{10} = \psi_0^{(N)},$$

that is, $\psi_0^{(N)} K_0^{(N)} = 0$. As pointed out above, we will not be able to solve this equation exactly. Nevertheless, for an appropriate choice of an approximate solution (see section 6.2 for more details), it can be proved that

- $\psi_n^{(N)}$ and $K_n^{(N)}$ converge for $N \to \infty$,
- $\psi_n = \lim\limits_{N\to\infty} \psi_n^{(N)}$ defines an invariant measure,
- $K_n = \lim\limits_{N\to\infty} K_n^{(N)}$ is invertible for $n \ge 1$ with $\psi_n = \psi_{n-1} p_{n-1,n} K_n$.
- $K_0 = \lim\limits_{N\to\infty} K_0^{(N)}$ has the eigenvalue 0, the corresponding eigenvector is unique up to constant multiples and can be chosen strictly positive.

The proofs in [BT95] for these statements are based on probabilistic interpretations of $K_n^{(N)}$ and $\left(K_n^{(N)}\right)^{-1}$ (to be precise, an interpretation for $R_n^{(N)} := p_{n,n+1} \left(K_{n+1}^{(N)}\right)^{-1}$ is given). Although we will prove the far more general Theorem 5.2.3 later on, we will give a direct proof of the interpretations (and existence) of $K_n^{(N)}$ and $\left(K_n^{(N)}\right)^{-1}$ for the case $d = 1$ in Lemma 2.1.1 below, since this interpretation is the crucial idea for finding a useful generalization of continued fractions. Before we do that, we summarize the algorithmic steps.

Algorithm 2.1.1. Let $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$ be block-tridiagonal and irreducible recurrent.

- Choose N large and set $R_{N-1}^{(N)} = p_{N-1,N}(I - p_{NN})^{-1}$.
- For $n = N, \dots, 1$ compute $R_{n-1}^{(N)} = p_{n-1,n}\left(I - p_{nn} - R_n^{(N)} p_{n+1,n}\right)^{-1}$.
- Compute an approximative non-trivial solution $\psi_0^{(N)}$ of $\psi_0^{(N)}\left(I - p_{00} - R_0^{(N)} p_{10}\right) = 0$.
- For $n = 0, \dots, N-1$ compute $\psi_{n+1}^{(N)} = \psi_n^{(N)} R_n^{(N)}$.

For the special case $d = 1$, we know the exact representation of the invariant measure (see section 1.1). Nevertheless, we remark that for $d = 1$, we may simply choose $\psi_0^{(N)} = 1$, and obtain convergence of $\psi^{(N)}$ to the invariant measure ψ with $\psi_0 = 1$. As a test, we can apply the algorithm to the introductory example presented in section 1.1.

n	exact solution ψ_n	forward computation	$\psi_n^{(N)}$ with $N = 100$
0	1	1	1
1	0.9	0.9	0.9
2	0.81	0.81	0.81
3	0.729	0.729	0.729
4	0.6561	0.6561	0.6561
5	0.5905	0.5905	0.5905
6	0.5314	0.5314	0.5314
7	0.4783	0.4783	0.4783
8	0.4305	0.4305	0.4305
10	0.3487	0.3487	0.3487
12	0.2824	0.2824	0.2824
15	0.2059	0.2066	0.2059
16	0.1853	0.1918	0.1853
17	0.1668	0.2250	0.1668
18	0.1501	0.6738	0.1501
19	0.1351	4.848	0.1351
20	0.1216	42.54	0.1216
25	0.07179	$2.505 \cdot 10^{6}$	0.07179
30	0.04239	$1.479 \cdot 10^{11}$	0.04239
40	0.01478	$5.157 \cdot 10^{20}$	0.01478
50	0.005154	$1.798 \cdot 10^{30}$	0.005154
90	$7.618 \cdot 10^{-5}$	$2.658 \cdot 10^{68}$	$7.618 \cdot 10^{-5}$
95	$4.498 \cdot 10^{-5}$	$1.569 \cdot 10^{73}$	$4.498 \cdot 10^{-5}$
98	$3.279 \cdot 10^{-5}$	$1.144 \cdot 10^{76}$	$3.276 \cdot 10^{-5}$
99	$2.951 \cdot 10^{-5}$	$1.030 \cdot 10^{77}$	$2.922 \cdot 10^{-5}$
100	$2.656 \cdot 10^{-5}$	$9.627 \cdot 10^{77}$	$2.391 \cdot 10^{-5}$

We see that the computation of $\psi_n^{(100)}$ is not effected by numerical instability. As n approaches N, we see slight errors which can be explained by the truncation procedure. For obtaining a more 'trustworthy' approximation for ψ_{99} or ψ_{100}, we should use a larger N.

Another effect of the algorithm is that $\psi_n^{(N)} \uparrow \psi_n$, and in particular, $\psi_n^{(N)} \leq \psi_n$. This fact can be proven by identifying $\psi^{(N)}$ as minimal subinvariant measure of $(p_{ij})_{i,j=0}^{N}$ (see section 6.2.1 for details). For $d = 1$, it is a direct consequence of $\left(K_n^{(N)}\right)^{-1}$ increasing monotonically in N, see Lemma 2.1.1 below.

Hence, for $f \geq 0$, monotone convergence yields $\psi^{(N)} f \uparrow \psi f$. For practical purposes, we need a **computable** upper bound for ψf, or equivalently, an estimate on the error $\psi f - \psi^{(N)} f$.

As pointed out above, we will see that we can represent ψf as a **generalized** continued fraction and $\psi^{(N)} f$ as an approximant. Furthermore, our definition of gcfs will be capable of dealing with arbitrarily structured transition probability matrices. As described above, we proceed with finding a probabilistic interpretation of $K_n^{(N)}$ and $\left(K_n^{(N)}\right)^{-1}$.

Lemma 2.1.1. *Let $(X_m)_{m\in\mathbb{N}_0}$ be a Markov chain in discrete time with (scalar) tridiagonal irreducible recurrent transition probability matrix $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$, and let $K_n^{(N)}$ be defined by*

$$\begin{aligned} K_N^{(N)} &= 1 - p_{NN}, \\ K_n^{(N)} &= 1 - p_{nn} - p_{n,n+1}\left(K_{n+1}^{(N)}\right)^{-1} p_{n+1,n}, \qquad 0 \leq n < N. \end{aligned}$$

a) Let $\tau_n = \inf\{m \geq 1 :\ X_m = n\}$ be the first hitting time for state n (where $\inf\emptyset = 0$), and let $T_{n,N} = \inf\{m \geq 1 :\ X_m \notin \{n, \ldots, N\}$ be the first time at which the Markov chain leaves $\{n, \ldots, N\}$. Then

$$K_n^{(N)} = 1 - \mathbb{P}\left(\tau_n < T_{n,N} | X_0 = n\right), \qquad n \leq N.$$

b) $K_n^{(N)} \neq 0$ for all $n \leq N$, and

$$\left(K_n^{(N)}\right)^{-1} = 1 + \mathbb{E}\left[\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m) | X_0 = n\right], \qquad n \leq N.$$

c) The limit $K_n := \lim\limits_{N\to\infty} K_n^{(N)} = 1 - \mathbb{P}(\tau_n < T_n | X_0 = n)$ exists for all $n \in \mathbb{N}_0$, where $T_n = \inf\{m > 0 :\ X_m < n\}$. In particular, $K_0 = 1 - \mathbb{P}(\tau_n < \infty | X_0 = n) = 0$.

d) $\left(K_n^{(N)}\right)^{-1}$ increases monotonically in N. For $n \geq 1$, we have

$$\lim_{N\to\infty}\left(K_n^{(N)}\right)^{-1} = K_n^{-1} = 1 + \mathbb{E}\left[\sum_{m=1}^{T_n} \mathbf{1}_{\{n\}}(X_m) | X_0 = n\right].$$

Proof. Additionally, we define $\tau_n^{(k)}$ to be the kth hitting time for state n, that is $\tau_n^{(1)} = \tau_n$ and

$$\tau_n^{(k)} = \inf\{m > \tau_n^{(k-1)} :\ X_m = n\}.$$

We have

$$\mathbb{P}(\tau_n^{(k)} < T_{n,N} | X_0 = n) = \left(\mathbb{P}\left(\tau_n < T_{n,N} | X_0 = n\right)\right)^k, \tag{2.1.4}$$

since for $k \geq 2$, Markov property and homogeneity yield

$$\begin{aligned}
\mathbb{P}(\tau_n^{(k)} < T_{n,N} | X_0 = n) &= \sum_{m=1}^{\infty} \mathbb{P}\left(\tau_n^{(k-1)} = m, m < \tau_n^{(k)} < T_{n,N} | X_0 = n\right) \\
&= \sum_{m=1}^{\infty} \mathbb{P}\left(\tau_n^{(k-1)} = m < T_{n,N} | X_0 = n\right) \\
&\qquad \cdot \mathbb{P}\left(m < \tau_n^{(k)} < T_{n,N} | X_0 = n, \tau_n^{(k-1)} = m, T_{n,N} > m\right) \\
&= \sum_{m=1}^{\infty} \mathbb{P}\left(\tau_n^{(k-1)} = m < T_{n,N} | X_0 = n\right) \cdot \mathbb{P}\left(\tau_n^{(1)} < T_{n,N} | X_0 = n\right) \\
&= \mathbb{P}\left(\tau_n^{(k-1)} < T_{n,N} | X_0 = n\right) \cdot \mathbb{P}\left(\tau_n < T_{n,N} | X_0 = n\right).
\end{aligned}$$

a) $K_N^{(N)} = 1 - p_{NN} = 1 - \mathbb{P}(\tau_N < T_{N,N} | X_0 = N)$ is obvious, since $\tau_N < T_{N,N}$ occurs if and only if $X_1 = N$. Now let $n < N$.

Since $\mathbf{P}$ is tridiagonal, $\tau_n < T_{n,N}$ is only possible if

- $X_1 = n$, this occurs with probability p_{nn}.
- $X_1 = n+1$ and $X_2 = n$, this occurs with probability $p_{n,n+1}p_{n+1,n}$.
- $X_1 = n+1$, and then, for some $k \in \mathbb{N}$, the Markov chain returns k times to $n+1$ before leaving $\{n+1, \ldots, N\}$, and after the kth return, it steps to state n. This occurs with probability

$$\sum_{k=1}^{\infty} p_{n,n+1} \mathbb{P}\left(\tau_{n+1}^{(k)} < T_{n+1,N} | X_0 = n+1\right) p_{n+1,n}.$$

Using (2.1.4), we obtain

$$\mathbb{P}(\tau_n < T_{n,N} | X_0 = n) = p_{nn} + p_{n,n+1} \left(\sum_{k=0}^{\infty} \left(\mathbb{P}(\tau_{n+1} < T_{n+1,N} | X_0 = n+1)\right)^k \right) p_{n+1,n}.$$

We have $\mathbb{P}(\tau_{n+1} < T_{n+1,N} | X_0 = n+1) < 1 - p_{n+1,n}$ and $p_{n+1,n} > 0$ (due to irreducibility), and therefore, the geometric series converges with

$$1 - \mathbb{P}(\tau_n < T_{n,N} | X_0 = n) = 1 - p_{nn} - p_{n,n+1} \left(1 - \mathbb{P}(\tau_{n+1} < T_{n+1,N} | X_0 = n+1)\right)^{-1} p_{n+1,n}.$$

By induction, we obtain the statement.

b) For $n > 0$, we have $K_n^{(N)} > 1 - p_{n,n-1} > 0$ due to irreducibility, and for $n = 0$, we can use

$$\begin{aligned}
K_0^{(N)} &= \mathbb{P}(T_{0,N} > \tau_0 | X_0 = x) \geq \mathbb{P}(X_1 = 1, \ldots, X_N = N, X_{N+1} = N+1 | X_0 = 0) \\
&= p_{01} p_{12} \cdot \ldots \cdot p_{N,N+1} > 0.
\end{aligned}$$

For deriving the stochastic interpretation of the inverse, we remark that $\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m)$ counts the number of returns to state n before the Markov chain leaves $\{n, \ldots, N\}$ for the

first time. Hence, the event $\tau_n^{(k)} < T_{n,N}$ occurs if and only if $\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m) \geq k$. Using the considerations in a), we obtain

$$\begin{aligned}\left(K_n^{(N)}\right)^{-1} &= 1 + \sum_{k=1}^{\infty} \mathbb{P}\left(\tau_n^{(k)} < T_{n,N} | X_0 = n\right) \\ &= 1 + \sum_{k=1}^{\infty} \mathbb{P}\left(\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m) \geq k | X_0 = n\right) \\ &= 1 + \mathbb{E}\left[\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m) | X_0 = n\right]\end{aligned}$$

c) For $N \to \infty$, we use $\{\tau_n < T_n\} = \bigcap_{N=n}^{\infty} \{\tau_n < T_{n,N}\}$, where $\{\tau_n < T_{n,N}\} \supset \{\tau_n < T_{n,N+1}\}$. Hence, $\mathbb{P}(\tau_n < T_{n,N}) \downarrow \mathbb{P}(\tau_n < T_n)$, and the interpretation of K_n follows immediately. For $n = 0$, we have $T_0 = \inf \emptyset = \infty$, and hence $K_0 = 1 - \mathbb{P}(\tau_0 < \infty | X_0 = 0) = 1 - 1 = 0$ due to recurrence.

d) Since $K_n^{(N)}$ decreases, $\left(K_n^{(N)}\right)^{-1}$ increases in N. For $n \geq 1$, we know that

$$K_n = 1 - \mathbb{P}(\tau_n < T_n | X_0 = n) = \mathbb{P}(T_n > \tau_n | X_0 = n) \geq p_{n,n-1} > 0.$$

Since $x \mapsto \frac{1}{x}$ is continuous in $x \neq 0$, we obtain $\lim_{N\to\infty} \left(K_n^{(N)}\right)^{-1} = K_n^{-1}$. Finally, $\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m)$ increases monotonically to $\sum_{m=1}^{T_n} \mathbf{1}_{\{n\}}(X_m)$, and the stochastic interpretation follows from monotone convergence.

□

Basically, the idea for defining generalized continued fractions is

- to find a recursion for $K_n^{(N)} = 1 - \mathbb{P}(\tau_n < T_{n,N})$ which holds for any structure of the transition probability matrix,
- and to use this recursion as a definition.

For irreducible Markov chains, convergence of $K = \lim_{N\to\infty} K_0^{(N)}$ is guaranteed. In order to obtain more general convergence criteria, we rewrite the interpretation of $K_n^{(N)}$ in such a way that it is applicable to situations where $\mathbf{P}$ is no stochastic matrix anymore: Since $\tau_n < T_{n,N}$

is equivalent to $X_1, \ldots, X_{m-1} \in \{n+1, \ldots, N\}$ and $X_m = n$ for some $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
\mathbb{P}(\tau_n < T_{n,N} | X_0 = n) &= \sum_{m \in \mathbb{N}} \mathbb{P}(X_1, \ldots, X_{m-1} \in \{n+1, \ldots, N\},\ X_m = n | X_0 = n) \\
&= \sum_{\substack{m \in \mathbb{N} \\ i_1, \ldots, i_{m-1} \in \{n+1, \ldots, N\} \\ i_0 = i_m = n}} \mathbb{P}(X_1 = i_1, \ldots, X_m = i_m | X_0 = i_0) \\
&= \sum_{\substack{m \in \mathbb{N} \\ i_1, \ldots, i_{m-1} \in \{n+1, \ldots, N\} \\ i_0 = i_m = n}} \prod_{r=1}^{m} p_{i_{r-1}, i_r}.
\end{aligned}$$

Similarly, we can deal with

$$\begin{aligned}
\mathbb{P}(\tau_n < T_n | X_0 = n) &= \sum_{m \in \mathbb{N}} \mathbb{P}(X_1, \ldots, X_{m-1} \geq n+1,\ X_m = n | X_0 = n) \qquad \text{and} \\
\mathbb{E}\left[\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m) | X_0 = n \right] &= \sum_{m \in \mathbb{N}} \mathbb{P}(X_1, \ldots, X_{m-1} \in \{n, \ldots, N\},\ X_m = n | X_0 = n).
\end{aligned}$$

Lemma 2.1.2. *Under the conditions of Lemma* 2.1.1*, we have*

a) $\mathbb{P}(\tau_n < T_{n,N} | X_0 = n) = \sum_{\substack{m \in \mathbb{N} \\ i_1, \ldots, i_{m-1} \in \{n+1, \ldots, N\} \\ i_0 = i_m = n}} \prod_{r=1}^{m} p_{i_{r-1}, i_r}$ *for all* $n, N \in \mathbb{N}_0$ *with* $n \leq N$.

b) $\mathbb{P}(\tau_0 < \infty | X_0 = 0) = \sum_{\substack{m \in \mathbb{N} \\ i_1, \ldots, i_{m-1} \geq n+1 \\ i_0 = i_m = 0}} \prod_{r=1}^{m} p_{i_{r-1}, i_r}$.

c) $\mathbb{E}\left[\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m) | X_0 = n \right] = \sum_{\substack{m \in \mathbb{N} \\ i_1, \ldots, i_{m-1} \in \{n, \ldots, N\} \\ i_0 = i_m = n}} \prod_{r=1}^{m} p_{i_{r-1}, i_r}$ *for alle* $n, N \in \mathbb{N}_0$ *with* $n \leq N$.

2.2 S-series

In section 2.1, we pointed out that for a Markov chain with tridiagonal transition probability matrix $\mathbf{P} = (p_{ij})_{i,j \in \mathbb{N}_0}$, we have

$$K_n^{(N)} = 1 - p_{nn} - p_{n,n+1} \left(K_{n+1}^{(N)} \right)^{-1} p_{n+1,n}, \qquad n \leq N,$$

if we set

$$K_n^{(N)} = 1 - \mathbb{P}(\tau_n < T_{n,N} | X_0 = n) = 1 - \sum_{\substack{m \in \mathbb{N} \\ i_1, \ldots, i_{m-1} \in \{n+1, \ldots, N\} \\ i_0 = i_m = n}} \prod_{r=1}^{m} p_{i_{r-1}, i_r},$$

see Lemma 2.1.1 and Lemma 2.1.2. Furthermore, we have the interpretation

$$\left(K_n^{(N)}\right)^{-1} = 1 + \mathbb{E}\left[\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m)|X_0 = n\right] = 1 + \sum_{\substack{m\in\mathbb{N} \\ i_1,\ldots,i_{m-1}\in\{n,\ldots,N\} \\ i_0=i_m=n}} \prod_{r=1}^{m} p_{i_{r-1},i_r}$$

in the context of Markov chains. Now, we allow arbitrary $p_{ij} \in \mathcal{R}$ and $p_{ij} \neq 0$ for $|i-j| \geq 2$. We will use the representation of $K_n^{(N)}$ as a series of products for deriving a recursion for $K_n^{(N)}$. Afterwards, we will use this recursion scheme for defining generalized continued fractions.

The remainder of this section is dedicated to the task of finding appropriate recursions for $K_n^{(N)}$. Since we have to deal with quite a lot 'sums of products', we introduce the notation of S-series.

Definition 2.2.1. Let $\mathbf{P} = (p_{mn})_{m,n\in\mathbb{N}_0}$ be an $\mathcal{R}$-valued matrix, $i, j \in \mathbb{N}_0$ and $A \subset \mathbb{N}_0$. Then we define

$$S(\mathbf{P}, i, j, A) := \sum_{\substack{\ell\in\mathbb{N} \\ i_0,\ldots,i_\ell\in\mathbb{N}_0 \\ i_0=i,\ i_\ell=j \\ i_1,\ldots,i_{\ell-1}\in A}} \prod_{r=1}^{\ell} p_{i_{r-1},i_r}.$$

Since we did not specify the order of summation, this definition only makes sense in case of unconditional convergence (or positive values p_{ij}, see section B.3 in the appendix and section 4.8). As pointed out above, for tridiagonal stochastic irreducible matrices $\mathbf{P}$, we have the continued fraction-type recursion

$$K_n^{(N)} = 1 - p_{nn} - p_{n,n+1}\left(K_{n+1}^{(N)}\right)^{-1} p_{n+1,n}$$

for $K_n^{(N)} = 1 - S(\mathbf{P}, n, n, \mathbb{N}_{n+1,N})$. In our Banach algebra $\mathcal{R}$, we replace 1 by the unity I, and aim for a recursion scheme for $I - S(\mathbf{P}, n, n, \mathbb{N}_{n+1,N})$ which holds for arbitrarily structured matrices $\mathbf{P}$.

Theorem 2.2.1. *Let $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$ be an $\mathcal{R}$-valued matrix. In case that the series on the right-hand side converge unconditionally, we have*

$$S(\mathbf{P}, n, n, \mathbb{N}_{n+1,N}) = p_{nn} + \sum_{m=n+1}^{N} p_{nm} S(\mathbf{P}, m, n, \mathbb{N}_{n+1,N}) \tag{2.2.1}$$

$$S(\mathbf{P}, n, n, \mathbb{N}_{n+1,N}) = p_{nn} + \sum_{m=n+1}^{N} S(\mathbf{P}, n, m, \mathbb{N}_{n+1,N}) p_{mn} \tag{2.2.2}$$

for all $n, N \in \mathbb{N}$, $n \leq N$,

$$(I - S(\mathbf{P}, n, n, \mathbb{N}_{n+1,N}))^{-1} = I + S(\mathbf{P}, n, n, \mathbb{N}_{n,N}) \tag{2.2.3}$$

for all $n, N \in \mathbb{N}$, $n \le N$,

$$S(\mathbf{P}, n, k, \mathbb{N}_{n,N}) = (I + S(\mathbf{P}, n, n, \mathbb{N}_{n,N})) \left(p_{nk} + \sum_{m=n+1}^{N} p_{nm} S(\mathbf{P}, m, k, \mathbb{N}_{n+1,N}) \right) \tag{2.2.4}$$

$$S(\mathbf{P}, k, n, \mathbb{N}_{n,N}) = \left(p_{kn} + \sum_{m=n+1}^{N} S(\mathbf{P}, k, m, \mathbb{N}_{n+1,N}) p_{mn} \right) (I + S(\mathbf{P}, n, n, \mathbb{N}_{n,N})) \tag{2.2.5}$$

for all $k, n, N \in \mathbb{N}$, $k < n \le N$,

$$S(\mathbf{P}, m, k, \mathbb{N}_{n,N}) = S(\mathbf{P}, m, k, \mathbb{N}_{n+1,N}) + S(\mathbf{P}, m, n, \mathbb{N}_{n+1,N}) S(\mathbf{P}, n, k, \mathbb{N}_{n,N}) \tag{2.2.6}$$

$$S(\mathbf{P}, k, m, \mathbb{N}_{n,N}) = S(\mathbf{P}, k, m, \mathbb{N}_{n+1,N}) + S(\mathbf{P}, k, n, \mathbb{N}_{n,N}) S(\mathbf{P}, n, m, \mathbb{N}_{n+1,N}) \tag{2.2.7}$$

for all $k, n, m, N \in \mathbb{N}$, $k < n < m \le N$, *and*

$$S(\mathbf{P}, n, n, \mathbb{N}_{n,\infty}) = \lim_{N \to \infty} S(\mathbf{P}, n, n, \mathbb{N}_{n,N}). \tag{2.2.8}$$

Proof. Using unconditional convergence (in the sense of Lemma A.1.1 e) and Theorem A.1.1), we derive

$$\begin{aligned}
S(\mathbf{P}, n, n, \mathbb{N}_{n+1,N}) &= \sum_{\substack{\ell \in \mathbb{N} \\ i_0, \ldots, i_\ell \in \mathbb{N}_0 \\ i_0 = n,\ i_\ell = n \\ i_1, \ldots, i_{\ell-1} \in \{n+1, \ldots, N\}}} \prod_{r=1}^{\ell} p_{i_{r-1}, i_r} \\
&= p_{nn} + \sum_{m=n+1}^{N} \sum_{\substack{\ell \ge 2 \\ i_0, \ldots, i_\ell \in \mathbb{N}_0 \\ i_0 = n,\ i_1 = m,\ i_\ell = n \\ i_2, \ldots, i_{\ell-1} \in \{n+1, \ldots, N\}}} \prod_{r=1}^{\ell} p_{i_{r-1}, i_r} \\
&= p_{nn} + \sum_{m=n+1}^{N} p_{nm} \sum_{\substack{\ell \ge 2 \\ i_1, \ldots, i_\ell \in \mathbb{N}_0 \\ i_1 = m,\ i_\ell = n \\ i_2, \ldots, i_{\ell-1} \in \{n+1, \ldots, N\}}} \prod_{r=2}^{\ell} p_{i_{r-1}, i_r} \\
&= p_{nn} + \sum_{m=n+1}^{N} p_{nm} \sum_{\substack{\ell \in \mathbb{N} \\ i_0, \ldots, i_\ell \in \mathbb{N}_0 \\ i_0 = m,\ i_\ell = n \\ i_1, \ldots, i_{\ell-1} \in \{n+1, \ldots, N\}}} \prod_{r=1}^{\ell} p_{i_{r-1}, i_r} \\
&= p_{nn} + \sum_{m=n+1}^{N} p_{nm} S(\mathbf{P}, m, n, \mathbb{N}_{n+1,N}),
\end{aligned}$$

that is, (2.2.1). By specifying $i_{\ell-1} = m$ instead of $i_1 = m$, (2.2.2) follows analogously.

For proving (2.2.3), we consider

$$\begin{aligned}
& S(\mathbf{P},n,n,\mathbb{N}_{n,N}) - S(\mathbf{P},n,n,\mathbb{N}_{n+1,N}) \\
= & \sum_{\substack{\ell\geq 2 \\ i_0,\ldots,i_\ell\in\mathbb{N}_0 \\ i_0=n,\ i_\ell=n \\ i_1,\ldots,i_{\ell-1}\in\{n,\ldots,N\} \\ i_k=n \text{ for some } k\in\{1,\ldots,\ell-1\}}} \prod_{r=1}^{\ell} p_{i_{r-1},i_r} = \sum_{\ell\in\mathbb{N}}\sum_{k=1}^{\ell-1} \sum_{\substack{\ell\geq 2 \\ i_0,\ldots,i_\ell\in\mathbb{N}_0 \\ i_0=n,\ i_k=n,\ i_\ell=n \\ i_1,\ldots,i_{k-1}\in\{n,\ldots,N\} \\ i_{k+1},\ldots,i_{\ell-1}\in\{n,\ldots,N\}}} \prod_{r=1}^{\ell} p_{i_{r-1},i_r} \\
= & \sum_{k=1}^{\infty}\sum_{\ell=k+1}^{\infty} \sum_{\substack{i_0,\ldots,i_k\in\mathbb{N}_0 \\ i_0=n,\ i_k=n \\ i_1,\ldots,i_{k-1}\in\{n,\ldots,N\}}} \prod_{r=1}^{k} p_{i_{r-1},i_r} \sum_{\substack{i_k,\ldots,i_\ell\in\mathbb{N}_0 \\ i_k=n,\ i_\ell=n \\ i_{k+1},\ldots,i_{\ell-1}\in\{n+1,\ldots,N\}}} \prod_{r=k+1}^{\ell} p_{i_{r-1},i_r} \\
= & \sum_{k\in\mathbb{N}} \sum_{\substack{i_0,\ldots,i_k\in\mathbb{N}_0 \\ i_0=n,\ i_k=n \\ i_1,\ldots,i_{k-1}\in\{n,\ldots,N\}}} \prod_{r=1}^{k} p_{i_{r-1},i_r} \cdot \sum_{\ell\in\mathbb{N}} \sum_{\substack{i_0,\ldots,i_\ell\in\mathbb{N}_0 \\ i_0=n,\ i_\ell=n \\ i_1,\ldots,i_{\ell-1}\in\{n+1,\ldots,N\}}} \prod_{r=1}^{\ell} p_{i_{r-1},i_r} \\
= & S(\mathbf{P},n,n,\mathbb{N}_{n,N}) S(\mathbf{P},n,n,\mathbb{N}_{n+1,N}),
\end{aligned}$$

yielding

$$(I + S(\mathbf{P},n,n,\mathbb{N}_{n,N}))\,(I - S(\mathbf{P},n,n,\mathbb{N}_{n+1,N})) = I.$$

Similarly, we derive

$$(I - S(\mathbf{P},n,n,\mathbb{N}_{n+1,N}))\,(I + S(\mathbf{P},n,n,\mathbb{N}_{n,N})) = I,$$

and together, we obtain (2.2.3).

Within the large parenthesis on the right-hand side of (2.2.4), with similar arguments as for proving (2.2.1), we have $S(\mathbf{P},n,k,\mathbb{N}_{n+1,N})$. Now, similarly to the proof of (2.2.3),

$$S(\mathbf{P},n,n,\mathbb{N}_{n,N}) S(\mathbf{P},n,k,\mathbb{N}_{n+1,N}) = S(\mathbf{P},n,k,\mathbb{N}_{n,N}) - S(\mathbf{P},n,k,\mathbb{N}_{n+1,N}),$$

and (2.2.4) can be obtained easily. The proof of (2.2.5) follows similar considerations.

The idea for proving (2.2.3) yield the similar formulas

$$\begin{aligned}
S(\mathbf{P},m,n,\mathbb{N}_{n+1,N}) S(\mathbf{P},n,k,\mathbb{N}_{n,N}) &= S(\mathbf{P},m,k,\mathbb{N}_{n,N}) - S(\mathbf{P},m,k,\mathbb{N}_{n+1,N}) \qquad \text{and} \\
S(\mathbf{P},k,n,\mathbb{N}_{n,N}) S(\mathbf{P},n,m,\mathbb{N}_{n+1,N}) &= S(\mathbf{P},k,m,\mathbb{N}_{n,N}) - S(\mathbf{P},k,m,\mathbb{N}_{n+1,N}),
\end{aligned}$$

and we obtain (2.2.6) and (2.2.7).

Finally, (2.2.8) directly follows from Lemma A.1.1. □

2.3 Definition of generalized continued fractions

Following the steps outlined at the beginning of this chapter, the recursion scheme for $I - S(\mathbf{P},n,n,\mathbb{N}_{n+1,N})$ enables us to define generalized continued fractions: We set $K_n^{(N)} = I - S(\mathbf{P},n,n,\mathbb{N}_{n+1,N})$ and $L_{m,n,k}^{(N)} = S(\mathbf{P},m,k,\mathbb{N}_{n,N})$ or $R_{k,n,m}^{(N)} = S(\mathbf{P},k,m,\mathbb{N}_{n,N})$, and we replace $b_n = I - p_{nn}$, $a_{mn} = -p_{mn}$, $c_{nm} = p_{nm}$ for $m < n$. This concept results in

Definition 2.3.1. Let $\mathbf{H}$ be an $\mathcal{R}$-valued infinite matrix

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & a_{02} & a_{03} & \cdots \\ -c_{10} & b_1 & a_{12} & a_{13} & \cdots \\ -c_{20} & -c_{21} & b_2 & a_{23} & \cdots \\ -c_{30} & -c_{31} & -c_{32} & b_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Starting with $n = N$ iterate

$$(2.3.1)\quad K_n^{(N)} = b_n + \sum_{m=n+1}^{N} a_{nm} L_{m,n+1,n}^{(N)} = b_n - \sum_{m=n+1}^{N} R_{n,n+1,m}^{(N)} c_{mn},$$

$$(2.3.2)\quad L_{n,n,k}^{(N)} = \left(K_n^{(N)}\right)^{-1} \left(c_{nk} - \sum_{m=n+1}^{N} a_{nm} L_{m,n+1,k}^{(N)}\right), \quad k = 0, \ldots, n-1,$$

$$(2.3.3)\quad L_{m,n,k}^{(N)} = L_{m,n+1,k}^{(N)} + L_{m,n+1,n}^{(N)} L_{n,n,k}^{(N)}, \quad k = 0, \ldots, n-1, \quad m = n+1, \ldots, N,$$

$$(2.3.4)\quad R_{k,n,n}^{(N)} = \left(-a_{kn} + \sum_{m=n+1}^{N} R_{k,n+1,m}^{(N)} c_{mn}\right) \left(K_n^{(N)}\right)^{-1}, \quad k = 0, \ldots, n-1,$$

$$(2.3.5)\quad R_{k,n,m}^{(N)} = R_{k,n+1,m}^{(N)} + R_{k,n,n}^{(N)} R_{n,n+1,m}^{(N)}, \quad k = 0, \ldots, n-1, \quad m = n+1, \ldots, N,$$

down to $n = 0$ (equation (2.3.1)) or $n = 1$ respectively (equations (2.3.2), (2.3.3), (2.3.4), (2.3.5)). If for almost all $N \in \mathbb{N}$, $K_0^{(N)}$ is well-defined (that is, $K_N^{(N)}, \ldots, K_1^{(N)} \in \mathcal{R}^*$), and

$$K = \lim_{N \to \infty} K_0^{(N)} \in \mathcal{R}$$

exists, K is said to be a *convergent generalized continued fraction*, abbreviated by

$$K = \operatorname{gcf}(\mathbf{H}).$$

$K_0^{(N)}$ is referred to as the Nth *approximant for the gcf* K.

Before we proceed with the derivation of convergence results, we give some remarks on the definition:

- For tridiagonal matrices, the recursion for $K_n^{(N)}$ results in

  $$K_n^{(N)} = b_n + a_{n,n+1} \left(K_{n+1}^{(N)}\right)^{(-1)} c_{n+1,n}.$$

  The letters b, a, c and the signs of a_{mn} and c_{nm} are chosen in such a way that they correspond to traditional definitions of continued fractions.

- The definition is inspired by the recursions which hold for the S-series in case of unconditional convergence. Note that the recursions in Definition 2.3.1 may yield well-defined approximants and convergent gcfs even in cases where the corresponding S-series do not converge (unconditionally).

- For tridiagonal $\mathbf{P}$, we have $R^{(N)}_{n,n+1,n+1} = -a_{n,n+1}\left(K^{(N)}_{n+1}\right)^{-1} = p_{n,n+1}\left(K^{(N)}_{n+1}\right)^{-1}$. In the context of computing invariant measures for Markov chains (see section 2.1), we referred to the limit as R_n, which corresponds to the usage of R_n in [Han99; BT95].

- In general, the matrices $R^{(N)}_{k,n,m}$ and their limits will be helpful to solve systems of the form $\psi\mathbf{H} = 0$. Conversely, the matrices $L^{(N)}_{m,n,k}$ and their limits will be useful for dealing with systems of the form $\mathbf{H}x = 0$. We will come back to this topic in chapter 4.

- $K^{(N)}_n$ is well-defined, that is, both recursions in (2.3.1) yield the same value $K^{(N)}_n$.

Lemma 2.3.1. *Define* $K^{(N)}_n, L^{(N)}_{m,n,k}, R^{(N)}_{k,m,n}$ *by*

$$K^{(N)}_n = b_n + \sum_{m=n+1}^{N} a_{nm} L^{(N)}_{m,n+1,n}$$

and equations (2.3.2), (2.3.3), (2.3.4), (2.3.5). *Then,*

$$K^{(N)}_n = b_n - \sum_{m=n+1}^{N} R^{(N)}_{n,n+1,m} c_{mn}, \qquad n \le N.$$

Proof. For $n = N$, the statement is trivial. In order to obtain it for $n < N$, we prove

$$(S_n) \qquad \sum_{m=n}^{N} a_{km} L^{(N)}_{m,n,\ell} = -\sum_{m=n}^{N} R^{(N)}_{k,n,m} c_{m\ell}, \qquad 0 \le k, \ell < n \le N$$

for $n = 1, \ldots, N$ by (backward) induction with respect to n, where (S_{n+1}) for $k = \ell = n$ yields the statement of the theorem.

(S_N) directly follows from

$$a_{kN} L^{(N)}_{NN\ell} = a_{kN} b_N^{-1} c_{N\ell} = -\left(a_{kN} b_N^{-1}\right) c_{N\ell} = -R^{(N)}_{kNN} c_{N\ell}.$$

Now, let (S_{n+1}) be true for some $n \in \{2, \dots, N\}$, and let $k, \ell < n$. Then,

$$\begin{aligned}
\sum_{m=n}^{N} a_{km} L^{(N)}_{mn\ell} &\overset{(2.3.3)}{=} \sum_{m=n+1}^{N} a_{km} L^{(N)}_{m,n+1,\ell} + \left(a_{kn} + \sum_{m=n+1}^{N} a_{km} L^{(N)}_{m,n+1,n} \right) L^{(N)}_{nn\ell} \\
&\overset{(2.3.2)}{=} \sum_{m=n+1}^{N} a_{km} L^{(N)}_{m,n+1,\ell} + \left(a_{kn} + \sum_{m=n+1}^{N} a_{km} L^{(N)}_{m,n+1,n} \right) \left(K^{(N)}_n \right)^{-1} \\
&\qquad \cdot \left(c_{n\ell} - \sum_{m=n+1}^{N} a_{nm} L^{(N)}_{m,n+1,\ell} \right) \\
&\overset{(S_{n+1})}{=} - \sum_{m=n+1}^{N} R^{(N)}_{k,n+1,m} c_{m\ell} + \left(a_{kn} - \sum_{m=n+1}^{N} R^{(N)}_{k,n+1,m} c_{mn} \right) \left(K^{(N)}_n \right)^{-1} \\
&\qquad \cdot \left(c_{n\ell} + \sum_{m=n+1}^{N} R^{(N)}_{n,n+1,m} c_{m\ell} \right) \\
&\overset{(2.3.4)}{=} - \sum_{m=n+1}^{N} R^{(N)}_{k,n+1,m} c_{m\ell} - R^{(N)}_{knn} \left(c_{n\ell} + \sum_{m=n+1}^{N} R^{(N)}_{n,n+1,m} c_{m\ell} \right) \\
&\overset{(2.3.5)}{=} - \sum_{m=n}^{N} R^{(N)}_{knm} c_{m\ell}.
\end{aligned}$$

that is, (S_n) holds. □

In chapter 3, we want to derive convergence criteria for $\operatorname{gcf}(\mathbf{H})$. As pointed out above, convergence of the S-series is no necessary condition for convergence of the gcf. Nevertheless, it is a sufficient condition, since in case of unconditional convergence of the S-series, we have

$$\begin{aligned}
K^{(N)}_n &= I - S(\mathbf{P}, n, n, \mathbb{N}_{n+1,N}), \\
K_n &= I - S(\mathbf{P}, n, n, \mathbb{N}_{n+1,\infty}), \\
L^{(N)}_{m,n,k} &= S(\mathbf{P}, m, k, \mathbb{N}_{n,N}), \\
R^{(N)}_{k,n,m} &= S(\mathbf{P}, k, m, \mathbb{N}_{n,N}).
\end{aligned}$$

Hence, we immediately obtain our first convergence criterion.

Theorem 2.3.1. *Let* $\mathbf{H}$ *as in Definition* 2.3.1*, and define* $\mathbf{P} = (p_{ij})_{i,j \in \mathbb{N}_0}$ *by* $p_{nn} = I - b_n$*,* $p_{mn} = -a_{mn}$*,* $p_{nm} = c_{nm}$ *for* $m < n$*.*

- *Let* $S(\mathbf{P}, 0, 0, \mathbb{N})$ *converge unconditionally,*
- *let* $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ *converge unconditionally for all* $n, N \in \mathbb{N}$*,* $n \leq N$*, and*
- *let* $S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ *converge unconditionally for all* $k, n, m, N \in \mathbb{N}_0$ *with* $k < n \leq m \leq N$ *or let* $S(\mathbf{P}, k, m, \mathbb{N}_{n,N})$ *converge unconditionally for all* $k, n, m, N \in \mathbb{N}_0$ *with* $k < n \leq m \leq N$

Then $\operatorname{gcf}(\mathbf{H})$ *converges with* $\operatorname{gcf}(\mathbf{H}) = I - S(\mathbf{P}, 0, 0, \mathbb{N})$*.*

Although it is an immediate consequence of the definition, the statement of Theorem 2.3.1 will turn out to be quite strong. In section 3.2, we will see that we are able to obtain Pringsheim-type convergence criteria for gcfs by means of Theorem 2.3.1. Furthermore, by construction, the criterion in Theorem 2.3.1 is met by many gcfs arising in the context of Markov chains, see chapter 5.

2.4 Literature review

To the knowledge of the author, there is no literature dealing with gcfs in the sense of Definition 2.3.1, terms as 'generalized continued fractions', 'matrix continued fractions' or 'continued fractions in Banach algebras' are used for special cases.

We will consider these special cases in the literature reviews at the end of the corresponding chapters, that is, in section 3.4, section 5.4, section 6.4 and section 7.4.

Chapter 3

Pringsheim-type convergence criteria for generalized continued fractions

In section 1.2, we mentioned that for continued fractions

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}$$

with coefficients $b_n, a_n \in \mathbb{C}$, Pringsheim's criterion guarantees convergence whenever the condition $|b_n| \geq |a_n| + 1$ is met for all $n \in \mathbb{N}$.

In this chapter, we intend to find Pringsheim-type convergence criteria for gcfs in the sense of Definition 2.3.1. In particular, we will obtain Pringsheim-type criteria for all special cases which we will discuss in section 7.4.

As a preparation for proving Pringsheim-type criteria, we need the concept of equivalence transformations.

3.1 Equivalence transformations for gcfs

A main issue of this thesis is the proof of convergence criteria for gcfs. Often, such criteria cannot be applied directly, but an appropriate modification of the coefficients may enable us to apply a convergence criterion. In this section, we will develop equivalence transformations, that is, modifications of the coefficients which do not have any influence on whether or not a gcf converges.

Define

$$\tilde{b}_n = \lambda_n b_n \rho_n^{-1},\ \tilde{a}_{nm} = \lambda_n a_{nm} \rho_m^{-1},\ \tilde{c}_{mn} = \lambda_m c_{mn} \rho_n^{-1}$$

with $\mathcal{R}^*$-valued sequences $(\lambda_n)_{n\in\mathbb{N}_0}$ and $(\rho_n)_{n\in\mathbb{N}_0}$. Furthermore, define $K_n^{(N)}$, $L_{m,n,k}^{(N)}$ by using b_n, a_{nm}, c_{mn} and define $\tilde{K}_n^{(N)}$ and $\tilde{L}_{m,n,k}^{(N)}$ by using $\tilde{b}_n$, $\tilde{a}_{nm}$ and $\tilde{c}_{mn}$. Then, by induction with

respect to $n = N, N-1, \ldots, 0$, it is seen that

$$\tilde{K}_n^{(N)} = \lambda_n K_n^{(N)} \rho_n^{-1}, \qquad \tilde{L}_{m,n,k}^{(N)} = \rho_m L_{m,n,k}^{(N)} \rho_k^{-1}, \qquad \tilde{R}_{k,n,m}^{(N)} = \lambda_k R_{k,n,m}^{(N)} \lambda_m^{-1}.$$

This relationship yields

Theorem 3.1.1. *Let*

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & a_{02} & a_{03} & \cdots \\ -c_{10} & b_1 & a_{12} & a_{13} & \cdots \\ -c_{20} & -c_{21} & b_2 & a_{23} & \cdots \\ -c_{30} & -c_{31} & -c_{32} & b_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad \textit{and} \quad \tilde{\mathbf{H}} = \begin{pmatrix} \tilde{b}_0 & \tilde{a}_{01} & \tilde{a}_{02} & \tilde{a}_{03} & \cdots \\ -\tilde{c}_{10} & \tilde{b}_1 & \tilde{a}_{12} & \tilde{a}_{13} & \cdots \\ -\tilde{c}_{20} & -\tilde{c}_{21} & \tilde{b}_2 & \tilde{a}_{23} & \cdots \\ -\tilde{c}_{30} & -\tilde{c}_{31} & -\tilde{c}_{32} & \tilde{b}_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where

$$\tilde{b}_n = \lambda_n b_n \rho_n^{-1}, \ \tilde{a}_{nm} = \lambda_n a_{nm} \rho_m^{-1}, \ \tilde{c}_{mn} = \lambda_m c_{mn} \rho_n^{-1}$$

with $\mathcal{R}^*$-valued sequences $(\lambda_n)_{n\in\mathbb{N}_0}$ and $(\rho_n)_{n\in\mathbb{N}_0}$. Then $\operatorname{gcf}(\mathbf{H})$ *converges if and only if* $\operatorname{gcf}\left(\tilde{\mathbf{H}}\right)$ *converges, and in case of convergence, we have*

$$\operatorname{gcf}\left(\tilde{\mathbf{H}}\right) = \lambda_0 \operatorname{gcf}(\mathbf{H}) \rho_0^{-1}.$$

3.2 Pringsheim-type convergence criteria

Non-generalized continued fractions in $\mathbb{C}$ are obtained from the general definition by putting $\mathcal{R} = \mathbb{C}$, $a_{mn} = c_{nm} = 0$ for $n \geq m+2$ and $c_{n,n-1} = 1$, yielding

$$\operatorname{gcf}(\mathbf{H}) = b_0 + \frac{a_{01}}{b_1 + \frac{a_{12}}{b_2 + \ddots}},$$

With this notation, Pringsheim [Pri99; Pri05] proved that

$$|b_n| \geq |a_{n-1,n}| + 1, \qquad n \geq 1$$

is a sufficient condition for convergence. In [Bau15], non-generalized continued fractions in Banach algebras $\mathcal{R}$ were considered, that is $a_{mn} = c_{nm} = 0$ for $n \geq m+2$, and it turned out that a sufficient criterion for convergence of $\operatorname{gcf}(\mathbf{H})$ is given by $a_{n-1,n} \neq 0$, $c_{n,n-1} \neq 0$ for all $n \in \mathbb{N}$ and

$$\begin{array}{rcll} \left|\left|b_n^{-1} a_{n,n+1}\right|\right| + \left|\left|b_n^{-1} c_{n,n-1}\right|\right| & \leq & 1, & n \in \mathbb{N} \quad \text{or} \\ \left|\left|c_{n+1,n} b_n^{-1}\right|\right| + \left|\left|a_{n-1,n} b_n^{-1}\right|\right| & \leq & 1, & n \in \mathbb{N}. \end{array}$$

In this section, we aim at finding similar results for gcfs. For this purpose, we make use of the following notations:

- Let $E \neq \emptyset$ be a discrete (finite or countably infinite) set and let $\mathbf{U} = (u_{ij})_{i,j\in E}$ be an arbitrary $\mathbb{R}_{\geq 0}$-valued matrix. For $i, j \in E$, we write $i \overset{\mathbf{U}}{\to} j$ if there is some $n \in \mathbb{N}_0$ such that $u_{ij}^{(n)} > 0$, where $\left(u_{ij}^{(n)}\right)_{i,j\in E} = \mathbf{U}^n$. We have $i \overset{\mathbf{U}}{\to} j$ if and only if there are $n \in \mathbb{N}$ and $i_0, i_1, \ldots, i_n \in E$ with $i_0 = i, i_n = j$ and $u_{i_{r-1},i_r} > 0$ for all $r = 0, \ldots, n-1$.
- For a matrix $\mathbf{U} = (u_{ij})_{i,j\in\mathbb{N}_0}$ with arbitrary values, we set

$$\mathbf{U}_{n,N} = (u_{ij})_{i,j=n}^{N} \qquad \text{and} \qquad \mathbf{U}_{n,\infty} = (u_{ij})_{i,j=n}^{\infty}.$$

3.2.1 Preparations

We will prove Pringsheim-type criteria for gcfs by means of Theorem 2.3.1. It turns out that we can replace the unconditional convergence of the series $S(\mathbf{P}, i, j, A)$ by the much more restrictive condition

$$S(\mathbf{P}, i, j, A) := \sum_{\substack{\ell\in\mathbb{N} \\ i_0,\ldots,i_\ell\in\mathbb{N}_0 \\ i_0=i,\ i_\ell=j \\ i_1,\ldots,i_{\ell-1}\in A}} \prod_{r=1}^{\ell} ||p_{i_{r-1},i_r}|| < \infty.$$

Theorem 3.2.1. *Let $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$, and let $\mathcal{N}(\mathbf{P}) = (||p_{ij}||)_{i,j\in\mathbb{N}_0}$. Furthermore, let $B \subset A \subset \mathbb{N}_0$, and $i, j \in \mathbb{N}_0$, and let $S(\mathcal{N}(\mathbf{P}), i, j, A)$ converge. Then $S(\mathbf{P}, i, j, A)$ converges absolutely (and thus, unconditionally) and*

$$||S(\mathbf{P}, i, j, A) - S(\mathbf{P}, i, j, B)|| \leq S(\mathcal{N}(\mathbf{P}), i, j, A) - S(\mathcal{N}(\mathbf{P}), i, j, B).$$

Proof. Absolute convergence of $S(\mathbf{P}, i, j, A)$, that is

$$\sum_{\substack{n\in\mathbb{N} \\ i_0,\ldots,i_n\in\mathbb{N}_0 \\ i_0=i,\ i_n=j \\ i_1,\ldots,i_{n-1}\in A}} \left|\left| \prod_{r=1}^{n} p_{i_{r-1},i_r} \right|\right| < \infty,$$

implies unconditional convergence of $S(\mathbf{P}, i, j, A)$ (see Theorem A.2.1). Due to submultiplicativity of $||\cdot||$, it is clear that

$$\sum_{\substack{n\in\mathbb{N} \\ i_0,\ldots,i_n\in\mathbb{N}_0 \\ i_0=i,\ i_n=j \\ i_1,\ldots,i_{n-1}\in A}} \prod_{r=1}^{n} ||p_{i_{r-1},i_r}|| < \infty$$

guarantees absolute convergence. The second statement is a direct consequence of submultiplicativity, too. □

The second part of this result is quite useful for discussing the speed of convergence of gcfs: Under the conditions of Theorem 2.3.1, for $i = j = 0$ and $A = \mathbb{N}$, $B = \mathbb{N}_{1,N}$, it turns out that

$$\begin{aligned}\left|\left|\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right|\right| &= ||S(\mathbf{P},0,0,\mathbb{N}) - S(\mathbf{P},0,0,\mathbb{N}_{1,N})|| \\ &\leq S(\mathcal{N}(\mathbf{P}),0,0,\mathbb{N}) - S(\mathcal{N}(\mathbf{P}),0,0,\mathbb{N}_{1,N}).\end{aligned}$$

For proving Pringsheim-type criteria, we make use of equivalence transformations. The coefficients λ_n, ρ_n are chosen such that b_n is transformed to I.

- Neither convergence properties of $\operatorname{gcf}(\mathbf{H})$ nor upper bounds for $\left|\left|\operatorname{gcf}(\mathbf{H}) - K_n^{(N)}\right|\right|$ depend on b_0. Since all statements in this section concern convergence or such estimates, we will assume $b_0 \in \mathcal{R}^*$ without loss of generality.

- We put $\rho_n = \lambda_n b_n$ for all $n \in \mathbb{N}$. Then $\mathbf{H}$ is transformed into

$$\tilde{\mathbf{H}} = \begin{pmatrix} I & \lambda_0 a_{01} b_1^{-1} \lambda_1^{-1} & \lambda_0 a_{02} b_2^{-1} \lambda_2^{-1} & \lambda_0 a_{03} b_3^{-1} \lambda_3^{-1} & \cdots \\ -\lambda_1 c_{10} \rho_0^{-1} & I & \lambda_1 a_{12} b_2^{-1} \lambda_2^{-1} & \lambda_1 a_{13} b_3^{-1} \lambda_3^{-1} & \cdots \\ -\lambda_2 c_{20} \rho_0^{-1} & -\lambda_2 c_{21} b_1^{-1} \rho_1^{-1} & I & \lambda_2 a_{23} b_3^{-1} \lambda_3^{-1} & \cdots \\ -\lambda_3 c_{30} \rho_0^{-1} & -\lambda_3 c_{31} b_1^{-1} \lambda_1^{-1} & -\lambda_3 c_{32} b_2^{-1} \lambda_2^{-1} & I & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

  We have $\operatorname{gcf}(\mathbf{H}) = \lambda_0 \operatorname{gcf}\left(\tilde{\mathbf{H}}\right) \rho_0^{-1} = \lambda_0 \left(I - S\left(\tilde{\mathbf{P}}, 0, 0, \mathbb{N}\right)\right) \rho_0^{-1}$ in case that

$$\tilde{\mathbf{P}} = \begin{pmatrix} 0 & -\lambda_0 a_{01} b_1^{-1} \lambda_1^{-1} & -\lambda_0 a_{02} b_2^{-1} \lambda_2^{-1} & -\lambda_0 a_{03} b_3^{-1} \lambda_3^{-1} & \cdots \\ -\lambda_1 c_{10} \rho_0^{-1} & 0 & -\lambda_1 a_{12} b_2^{-1} \lambda_2^{-1} & -\lambda_1 a_{13} b_3^{-1} \lambda_3^{-1} & \cdots \\ -\lambda_2 c_{20} \rho_0^{-1} & -\lambda_2 c_{21} b_1^{-1} \rho_1^{-1} & 0 & -\lambda_2 a_{23} b_3^{-1} \lambda_3^{-1} & \cdots \\ -\lambda_3 c_{30} \rho_0^{-1} & -\lambda_3 c_{31} b_1^{-1} \lambda_1^{-1} & -\lambda_3 c_{32} b_2^{-1} \lambda_2^{-1} & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

  satisfies the conditions of Theorem 2.3.1.

- Now take Theorem 3.2.1 into account. If suffices to prove that

$$\hat{\mathbf{P}} := \mathcal{N}\left(\tilde{\mathbf{P}}\right) = \begin{pmatrix} 0 & \left|\left|\lambda_0 a_{01} b_1^{-1} \lambda_1^{-1}\right|\right| & \left|\left|\lambda_0 a_{02} b_2^{-1} \lambda_2^{-1}\right|\right| & \left|\left|\lambda_0 a_{03} b_3^{-1} \lambda_3^{-1}\right|\right| & \cdots \\ \left|\left|\lambda_1 c_{10} \rho_0^{-1}\right|\right| & 0 & \left|\left|\lambda_1 a_{12} b_2^{-1} \lambda_2^{-1}\right|\right| & \left|\left|\lambda_1 a_{13} b_3^{-1} \lambda_3^{-1}\right|\right| & \cdots \\ \left|\left|\lambda_2 c_{20} \rho_0^{-1}\right|\right| & \left|\left|\lambda_2 c_{21} b_1^{-1} \lambda_1^{-1}\right|\right| & 0 & \left|\left|\lambda_2 a_{23} b_3^{-1} \lambda_3^{-1}\right|\right| & \cdots \\ \left|\left|\lambda_3 c_{30} \rho_0^{-1}\right|\right| & \left|\left|\lambda_3 c_{31} b_1^{-1} \lambda_1^{-1}\right|\right| & \left|\left|\lambda_3 c_{32} b_2^{-1} \lambda_2^{-1}\right|\right| & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

  satisfies the conditions of Theorem 2.3.1. In this case, Theorem 3.2.1 guarantees convergence of $\operatorname{gcf}(\mathbf{H})$ with

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq \hat{K}_0^{(N)} - \operatorname{gcf}\left(\hat{\mathbf{H}}\right) = S\left(\hat{\mathbf{P}}, 0, 0, \mathbb{N}\right) - S\left(\hat{\mathbf{P}}, 0, 0, \mathbb{N}_{1,N}\right),$$

where

$$\hat{\mathbf{H}} = \begin{pmatrix} 1 & -\left|\left|\lambda_0 a_{01} b_1^{-1} \lambda_1^{-1}\right|\right| & -\left|\left|\lambda_0 a_{02} b_2^{-1} \lambda_2^{-1}\right|\right| & -\left|\left|\lambda_0 a_{03} b_3^{-1} \lambda_3^{-1}\right|\right| & \cdots \\ \left|\left|\lambda_1 c_{10} \rho_0^{-1}\right|\right| & 1 & -\left|\left|\lambda_1 a_{12} b_2^{-1} \lambda_2^{-1}\right|\right| & -\left|\left|\lambda_1 a_{13} b_3^{-1} \lambda_3^{-1}\right|\right| & \cdots \\ \left|\left|\lambda_2 c_{20} \rho_0^{-1}\right|\right| & \left|\left|\lambda_2 c_{21} b_1^{-1} \lambda_1^{-1}\right|\right| & 1 & -\left|\left|\lambda_2 a_{23} b_3^{-1} \lambda_3^{-1}\right|\right| & \cdots \\ \left|\left|\lambda_3 c_{30} \rho_0^{-1}\right|\right| & \left|\left|\lambda_3 c_{31} b_1^{-1} \lambda_1^{-1}\right|\right| & \left|\left|\lambda_3 c_{32} b_2^{-1} \lambda_2^{-1}\right|\right| & 1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

- Due to commutativity of $\mathbb{R}$, we have $S\left(\hat{\mathbf{P}}^T, m, k, \mathbb{N}_{n,N}\right) = S\left(\hat{\mathbf{P}}, k, m, \mathbb{N}_{n,N}\right)$. Due to the symmetry of the statement, it suffices to prove that $\hat{\mathbf{P}}$ **or** $\hat{\mathbf{P}}^T$ satisfies the conditions of Theorem 2.3.1.

By summarizing we get

Theorem 3.2.2. *Let* $\mathbf{U} \in \left\{\hat{\mathbf{P}}, \hat{\mathbf{P}}^T\right\}$ *where* $\hat{\mathbf{P}}$ *is defined as above. If* $\mathbf{U}$ *satisfies the conditions of Theorem* 2.3.1*,* $\operatorname{gcf}(\mathbf{H})$ *converges with*

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq S(\mathbf{U}, 0, 0, \mathbb{N}) - S(\mathbf{U}, 0, 0, \mathbb{N}_{1,N}).$$

3.2.2 A general Pringsheim-type criterion

Now, in principle, we are ready for stating a very general Pringsheim-type criterion. In order to state the result in a succinct way, we introduce another notation.

Definition 3.2.1. Let $\mathbf{U} = (u_{ij})_{i,j \in \mathbb{N}_0}$ be a $\mathbb{R}_{\geq 0}$-valued matrix. A function $g : \mathbb{N}_0 \to \mathbb{R}$, $j \mapsto g_j$ is said to be a *Pringsheim-Lyapunov function* for $\mathbf{U}$ if

(PL0) $g_j > 0$ for all $j \in \mathbb{N}_0$ and $g_0 = 1$.

(PL1) $\sum_{k=0}^{\infty} u_{0k} g_k < \infty$.

(PL2) $\sum_{k=0}^{\infty} u_{mk} g_k \leq g_m$ for all $m \in \mathbb{N}$.

(PL3) for all $n, i, N \in \mathbb{N}$ with $n \leq i \leq N$ there is some $j \in \{n, \ldots, N\}$ with

- $i \xrightarrow{\mathbf{U}} j$ and
- $\sum_{k=n}^{N} u_{jk} g_k < g_j$.

Theorem 3.2.3. *Let* $\mathbf{U} \in \left\{\hat{\mathbf{P}}, \hat{\mathbf{P}}^T\right\}$ *where* $\hat{\mathbf{P}}$ *is defined as above, and let* g *be a Pringsheim-Lyapunov function for* $\mathbf{U}$*. Then* $\operatorname{gcf}(\mathbf{H})$ *converges with*

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq \sum_{m=N+1}^{\infty} S(\mathbf{U}, 0, m, \mathbb{N}_{1,N}) g_m.$$

Proof. If $S(\mathbf{U},0,0,\mathbb{N})$, $S(\mathbf{U},n,n,\mathbb{N}_{n,N})$ and $S(\mathbf{U},m,k,\mathbb{N}_{n,N})$ are convergent for $k,n,m,N\in\mathbb{N}$ with $k<n\le m\le N$, Theorem 3.2.2 guarantees convergence of gcf (**H**). We introduce the notation

$$S_L(\mathbf{U},i,j,A):=\sum_{\substack{1\le\ell\le L\\ i_0,\dots,i_\ell\in\mathbb{N}_0\\ i_0=i,\ i_\ell=j\\ i_1,\dots,i_{\ell-1}\in A}}\prod_{r=1}^{\ell}u_{i_{r-1},i_r},$$

and due to $u_{ij}\in[0,\infty)$, $S_L(\mathbf{U},i,j,A)$ converges monotonically increasing to $S(\mathbf{U},i,j,A)$.

- We prove $S_L(\mathbf{U},m,k,\mathbb{N}_{n,N})g_k\le g_m$ by induction. Obviously,

  $$S_1(\mathbf{U},m,k,\mathbb{N}_{n,N})g_k=u_{mk}g_k\le g_m.$$

  Let $L\ge 2$ and assume that the statement holds for $L-1$. Then, using (PL2), we obtain

  $$\begin{aligned}S_L(\mathbf{U},m,k,\mathbb{N}_{n,N})g_k &= u_{mk}g_k+\sum_{j=n}^{N}u_{mj}S_{L-1}(\mathbf{U},j,k,\mathbb{N}_{n,N})g_k\\ &\le u_{mk}g_k+\sum_{j=n}^{N}u_{mj}g_j\le g_m.\end{aligned}$$

  Since the estimate does not depend on L, we obtain $S(\mathbf{U},m,k,\mathbb{N}_{n,N})g_k\le g_m<\infty$.

- Similarly, for $n\le m\le N$, we have

  $$S_L(\mathbf{U},m,n,\mathbb{N}_{n+1,N})g_n \le u_{mn}g_n+\sum_{k=n+1}^{N}u_{mk}g_k=\sum_{k=n}^{N}u_{mk}g_k \qquad \text{for all } L\in\mathbb{N}$$

  $$\Rightarrow\qquad S(\mathbf{U},m,n,\mathbb{N}_{n+1,N})g_n \le \sum_{k=n}^{N}u_{mk}g_k$$

  for all $m\in\{n,\dots,N\}$. Now choose $j\in\{n,\dots,N\}$ acccording to (PL3). Then $S(\mathbf{U},j,n,\mathbb{N}_{n+1,N})g_n<g_j$. Next, let $i\in\{n,\dots,N\}$ with $u_{ij}>0$. Then

  $$S(\mathbf{U},i,n,\mathbb{N}_{n+1,N})g_n=u_{in}g_n+\sum_{k=n+1}^{N}u_{ik}S(\mathbf{U},k,n,\mathbb{N}_{n+1,N})g_n<\sum_{k=n}^{N}u_{ik}g_k\le g_i.$$

  By iterating this argument finitely many times (note that we have $n\xrightarrow{\mathbf{U}}j$), we obtain $S(\mathbf{U},n,n,\mathbb{N}_{n+1,N})g_n<g_n$, that is $S(\mathbf{U},n,n,\mathbb{N}_{n+1,N})<1$. Hence,

  $$S(\mathbf{U},n,n,\mathbb{N}_{n,N})=\frac{1}{1-S(\mathbf{U},n,n,\mathbb{N}_{n+1,N})}-1<\infty.$$

- The inequality $S(\mathbf{U},m,k,\mathbb{N}_{n,N})g_k\le g_m$ did not depend on n or N. Thus, for $N\to\infty$, we obtain $S(\mathbf{U},m,k,\mathbb{N}_{n,\infty})g_k\le g_m$, and in particular, $S(\mathbf{U},m,0,\mathbb{N})\le g_m$. Hence,

  $$S(\mathbf{U},0,0,\mathbb{N})=u_{00}+\sum_{m=0}^{\infty}u_{0m}S(\mathbf{U},m,0,\mathbb{N})\le\sum_{m=0}^{\infty}u_{0m}g_m<\infty$$

  due to (PL1).

Theorem 3.2.2 guarantees convergence of gcf (**H**) with

$$\begin{aligned}\left|\left|\lambda_0\left(\operatorname{gcf}(\mathbf{H})-K_0^{(N)}\right)\rho_0^{-1}\right|\right| &\leq S(\mathbf{U},0,0,\mathbb{N})-S(\mathbf{U},0,0,\mathbb{N}_{1,N})\\ &= \sum_{m=N+1}^{\infty} S(\mathbf{U},0,m,\mathbb{N}_{1,N})S(\mathbf{U},m,0,\mathbb{N})\\ &\leq \sum_{m=N+1}^{\infty} S(\mathbf{U},0,m,\mathbb{N}_{1,N})g_m.\end{aligned}$$

□

Condition (PL3) guarantees that the gcf does not degenerate. We give two easy criteria.

Theorem 3.2.4. *Let $\mathbf{U}=(u_{ij})_{i,j\in\mathbb{N}_0}$ be a $\mathbb{R}_{\geq 0}$-valued matrix meeting, and let g meet (PL0), (PL1) and (PL2). If*

(PL3a) for all $n\in\mathbb{N}$ there is some $m<n$ with $u_{nm}>0$ or
(PL3b) for all $n\in\mathbb{N}$ there is some $m>n$ with $u_{nm}>0$,

g is a Pringsheim-Lyapunov function.

Proof. Let (PL3a) hold. We prove (PL3) by induction with respect to i. Due to (PL2), $u_{nm}>0$ and $g_m>0$, we have $\sum_{k=n}^{N} u_{nk}g_k<g_n$. Now assume that (PL3) holds for all indices $<i$. If $u_{im}>0$ for $m<n$, again, (PL2) directly yields $\sum_{k=n}^{N} u_{ik}g_k<g_i$. If $u_{im}>0$ for $m\in\{n,\ldots,i-1\}$, the induction hypothesis applies.
If (PL3b) holds, we perform the same induction starting with $i=N$. □

We conclude the section with stating the result for the special case $g=1$.

Theorem 3.2.5. *Let $b_n\in\mathcal{R}^*$ for all $n\in\mathbb{N}$, let $(\lambda_n)_{n\in\mathbb{N}_0}$ be a $\mathcal{R}^*$-valued sequence, let*

$$\sum_{m=0}^{n-1}\left|\left|\lambda_m a_{mn} b_n^{-1}\lambda_n^{-1}\right|\right|+\sum_{m=n+1}^{\infty}\left|\left|\lambda_m c_{mn} b_n^{-1}\lambda_n^{-1}\right|\right|\leq 1 \tag{3.2.1}$$

for all $n\in\mathbb{N}$, let

$$\sum_{m=1}^{\infty}\left|\left|\lambda_m c_{m0}\rho_0^{-1}\right|\right|\leq 1 \tag{3.2.2}$$

for some $\rho_0 \in \mathcal{R}^*$, and let states $1, \ldots, N$ be transient (see appendix C) with respect to the substochastic matrix $\hat{\mathbf{P}}_{1,N}^T$. Then, $\operatorname{gcf}(\mathbf{H})$ *converges and for its approximants $K_0^{(N)}$, we have*

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq \hat{K}_0^{(N)} - \operatorname{gcf}\left(\hat{\mathbf{H}}\right) \tag{3.2.3}$$

$$= S\left(\hat{\mathbf{P}}, 0, 0, \mathbb{N}\right) - S\left(\hat{\mathbf{P}}, 0, 0, \mathbb{N}_{1,N}\right) \tag{3.2.4}$$

$$\leq \sum_{m=N+1}^{\infty} S\left(\hat{\mathbf{P}}^T, 0, m, \mathbb{N}_{1,N}\right) \tag{3.2.5}$$

$$= \sum_{m=N+1}^{\infty} S\left(\hat{\mathbf{P}}, m, 0, \mathbb{N}_{1,N}\right), \tag{3.2.6}$$

for all $N \in \mathbb{N}_0$, where $\hat{K}_0^{(N)}$ are the approximants of $\operatorname{gcf}\left(\hat{\mathbf{H}}\right)$*. In particular,*

$$\left|\left|\lambda_0(\operatorname{gcf}(\mathbf{H}) - b_0)\rho_0^{-1}\right|\right| \leq 1. \tag{3.2.7}$$

Sufficient criteria for transience of states $1, \ldots, N$ with respect to $\hat{\mathbf{P}}_{1,N}^T$ are given by

- *for all $n \in \mathbb{N}$ there is some $m > n$ with $a_{nm} \neq 0$ or*
- *for all $n \in \mathbb{N}$ there is some $m < n$ with $c_{nm} \neq 0$.*

Proof. Set $\mathbf{U} = \hat{\mathbf{P}}^T$ and $g_n = 1$ in Theorem 3.2.3. Then, (3.2.1) yields (PL2) and (3.2.2) yields (PL1) with

$$\sum_{m=0}^{\infty} u_{0m} g_m = \sum_{m=0}^{\infty} u_{0m} \leq 1.$$

The transience condition corresponds to (PL3), and the criteria for transience correspond to (PL3a) and (PL3b) respectively. The speed-of-convergence statements are consequences of those in Theorem 3.2.2 and Theorem 3.2.3. For $N = 0$, we obtain

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - b_0\right) \rho_0^{-1}\right|\right| \leq \sum_{m=1}^{\infty} S(\mathbf{U}, 0, m, \emptyset) = \sum_{m=0}^{\infty} u_{0m} \leq 1,$$

that is, (3.2.7). □

3.3 Pringsheim-type criteria: Special cases

In this section, we want to consider special cases for the structure of $\mathbf{H}$ and we will state simplified Pringsheim-type criteria. Still, we assume that $\hat{\mathbf{P}}$ is built up as in section 3.2.

3.3.1 Upper Hessenberg matrices

Let $c_{nm} = 0$ for $m \leq n-2$. Then we obtain an upper Hessenberg matrix

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & a_{02} & a_{03} & \cdots \\ -c_{10} & b_1 & a_{12} & a_{13} & \cdots \\ & -c_{21} & b_2 & a_{23} & \cdots \\ & & -c_{32} & b_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

In this case, the results of Theorem 3.2.3 and Theorem 3.2.5 simplify to

Theorem 3.3.1. *Let $\mathbf{H}$ be an upper Hessenberg matrix with $b_n \in \mathcal{R}^*$ and $c_{n+1,n} \neq 0$ for all $n \in \mathbb{N}$, let $(\lambda_n)_{n\in\mathbb{N}_0}$ be an $\mathcal{R}^*$-valued sequences, and let $\rho_0 \in \mathcal{R}^*$.*

- *Let $g : \mathbb{N}_0 \to \mathbb{R}_{>0}$, $n \mapsto g_n$ with $g_0 = 1$ and*

$$\sum_{k=0}^{m+1} g_k \hat{p}_{km} \leq g_m$$

for all $m \in \mathbb{N}$. Then, $\operatorname{gcf}(\mathbf{H})$ converges, and for its value, we have

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq g_{N+1} \hat{p}_{N+1,N} S\left(\hat{\mathbf{P}}, N, 0, \mathbb{N}_{1,N}\right)$$

for $N \geq 1$ and $\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - b_0\right) \rho_0^{-1}\right|\right| \leq g_1 \hat{p}_{10}$.

- *Let $\left|\left|\lambda_1 c_{10} \rho_0^{-1}\right|\right| \leq 1$, and let*

$$\sum_{m=0}^{n-1} \left|\left|\lambda_m a_{mn} b_n^{-1} \lambda_n^{-1}\right|\right| + \left|\left|\lambda_{n+1} c_{n+1,n} b_n^{-1} \lambda_n^{-1}\right|\right| \leq 1, \qquad n \in \mathbb{N}.$$

Then, $\operatorname{gcf}(\mathbf{H})$ converges, and for its value, we have $\left|\left|\lambda_0 (\operatorname{gcf}(\mathbf{H}) - b_0) \rho_0^{-1}\right|\right| \leq 1$.

Proof. We prove that the conditions of Theorem 3.2.3 or Theorem 3.2.5 respectively are satisfied.

- The conditions directly imply that g meets (PL0) and (PL2) for $\hat{\mathbf{P}}^T$. Furthermore, $\hat{\mathbf{P}}^T$ is a lower Hessenberg matrix, and hence,(PL1) is satisfied automatically. Finally, $c_{n+1,n} \neq 0$ implies (PL3b), and thus, (PL3) for $\hat{\mathbf{P}}^T$. Finally, the structure of Hessenberg matrices yields

$$\begin{aligned} S\left(\hat{\mathbf{P}}^T, 0, N+1, \mathbb{N}_{1,N}\right) &= S\left(\hat{\mathbf{P}}, N+1, 0, \mathbb{N}_{1,N}\right) \\ &= \hat{p}_{N+1,N} S\left(\hat{\mathbf{P}}, N, 0, \mathbb{N}_{1,N}\right). \end{aligned}$$

- (3.2.1) and (3.2.2) are obviously fulfilled for $\hat{\mathbf{P}}^T$. The transience condition is satisfied due to $c_{n+1,n} \neq 0$ (see last criterion in Theorem 3.2.5).

□

3.3.2 Lower Hessenberg matrices

Now, we consider the case that $a_{nm} = 0$ for $m \geq n+2$, that is $\mathbf{H}$ is a lower Hessenberg matrix

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & & & \\ -c_{10} & b_1 & a_{12} & & \\ -c_{20} & -c_{21} & b_2 & a_{23} & \\ -c_{30} & -c_{31} & -c_{32} & b_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

For this special case, we just state the general theorem involving a Lyapunov function, the proof is similar to that of Theorem 3.3.1.

Theorem 3.3.2. *Let* $\mathbf{H}$ *be a lower Hessenberg matrix with* $b_n \in \mathcal{R}^*$ *and* $a_{n,n+1} \neq 0$ *for all* $n \in \mathbb{N}$*, let* $(\lambda_n)_{n\in\mathbb{N}_0}$ *be an* $\mathcal{R}^*$*-valued sequence, let* $\rho_0 \in \mathcal{R}^*$*, let* $g : \mathbb{N}_0 \to \mathbb{R}_{>0}$*,* $n \mapsto g_n$ *with* $g_0 = 1$ *and*

$$\sum_{k=0}^{m+1} \hat{p}_{mk} g_k \leq g_m$$

for all $m \in \mathbb{N}$*. Then,* $\operatorname{gcf}(\mathbf{H})$ *converges, and for its value, we have*

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq S\left(\hat{\mathbf{P}}, 0, N, \mathbb{N}_{1,N}\right) \hat{p}_{N,N+1} g_{N+1}$$

for $N \geq 1$ *and* $\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - b_0\right) \rho_0^{-1}\right|\right| \leq \hat{p}_{01} g_1$*.*

3.3.3 Tridiagonal matrices

Finally, we consider the case of tridiagonal

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & & & \\ -c_{10} & b_1 & a_{12} & & \\ & -c_{21} & b_2 & a_{23} & \\ & & -c_{32} & b_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Here, $\operatorname{gcf}(\mathbf{H})$ is a non-generalized continued fraction, that is

$$\operatorname{gcf}(\mathbf{H}) = b_0 + a_{01} \left(b_1 + a_{12} \left(b_2 + \ldots\right)^{-1} c_{21}\right)^{-1} c_{10}.$$

Naturally, tridiagonal matrices are upper Hessenberg and lower Hessenberg matrices at the same time. Note that in this case, we can replace $c_{n+1,n} \neq 0$ by $a_{n,n+1} \neq 0$ in Theorem 3.3.1 (using (PL3a) instead of (PL3b)), and similarly, we can replace $a_{n,n+1} \neq 0$ by $c_{n+1,n} \neq 0$ in Theorem 3.3.2.

Theorem 3.3.3. *Let* $\mathbf{H}$ *be tridiagonal, let* $b_n^{-1} \in \mathcal{R}^*$ *for all* $n \in \mathbb{N}$ *and let*

- $a_{n,n+1} \neq 0$ *for all* $n \in \mathbb{N}_0$ *or*
- $c_{n+1,n} \neq 0$ *for all* $n \in \mathbb{N}_0$.

Furthermore, let $\mathbf{U} \in \left\{\hat{\mathbf{P}}, \hat{\mathbf{P}}^T\right\}$, *let* $(\lambda_n)_{n\in\mathbb{N}_0}$ *be an* $\mathcal{R}^*$*-valued sequence, let* $\rho_0 \in \mathcal{R}^*$, *and let*

$$u_{n,n-1}g_{n-1} - g_n + u_{n,n+1}g_{n+1} \leq 0$$

for some function $g : \mathbb{N}_0 \to \mathbb{R}_{>0}$ *with* $g_0 = 1$. *Then,* $\operatorname{gcf}(\mathbf{H})$ *converges, and for its value, we have*

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq \frac{u_{01}u_{12}\dots u_{N-1,N}u_{N,N+1}}{B_N} \cdot g_{N+1} \tag{3.3.1}$$

for all $N \in \mathbb{N}_0$, *where* B_n *is defined by* $B_0 = B_1 = 1$ *and* $B_n = B_{n-1} - u_{n-1,n}u_{n,n-1}B_{n-2}$ *for* $n \geq 2$.

Proof. Convergence of $\operatorname{gcf}(\mathbf{H})$ is implied by Theorem 3.3.1 and Theorem 3.3.2. The error estimate is obvious for $N = 0$, and for $N \geq 1$, we have to prove that

$$S(\mathbf{U}, 0, N, \mathbb{N}_{1,N}) = \frac{u_{01}\dots u_{N-1,N}}{B_N}. \tag{3.3.2}$$

First, we prove

$$1 + S(\mathbf{U}, N, N, \mathbb{N}_{1,N}) = \frac{B_{N-1}}{B_N} \tag{3.3.3}$$

by induction. For $N = 1$, (3.3.3) is a consequence of $S(\mathbf{U}, 1, 1, \{1\}) = 0$ and $B_0 = B_1 = 1$. Now let (3.3.3) be true for $N - 1$. Then,

$$\begin{aligned}
1 + S(\mathbf{U}, N, N, \mathbb{N}_{1,N}) &= \frac{1}{1 - S(\mathbf{U}, N, N, \mathbb{N}_{1,N-1})} \\
&= \frac{1}{1 - u_{N,N-1}S(\mathbf{U}, N-1, N-1, \mathbb{N}_{1,N-1})u_{N-1,N}} \\
&= \frac{1}{1 - u_{N-1,N}u_{N,N-1}\frac{B_{N-2}}{B_{N-1}}} \\
&= \frac{B_{N-1}}{B_{N-1} - u_{N-1,N}u_{N,N-1}B_{N-2}} = \frac{B_{N-1}}{B_N},
\end{aligned}$$

which provides the result for N. (3.3.2) can be proved similarly: For $N = 1$, $S(\mathbf{U}, 0, 1, \{1\}) = \frac{u_{01}}{1}$. Now, let (3.3.2) be given for $N - 1$. Then,

$$\begin{aligned}
S(\mathbf{U}, 0, N, \mathbb{N}_{1,N}) &= S(\mathbf{U}, 0, N-1, \mathbb{N}_{1,N-1})u_{N-1,N}(1 + S(\mathbf{U}, N, N, \mathbb{N}_{1,N})) \\
&= \frac{u_{01}\dots u_{N-2,N-2}}{B_{N-1}}u_{N-1,N}\frac{B_{N-1}}{B_N} \\
&= \frac{u_{01}\dots u_{N-1,N}}{B_N},
\end{aligned}$$

as claimed. □

For $g_n = 1$, we obtain the following result.

Theorem 3.3.4. *Let* $\mathbf{H}$ *be tridiagonal, let* $b_n^{-1} \in \mathcal{R}^*$ *for all* $n \in \mathbb{N}$*, let*

- $a_{n,n+1} \neq 0$ *for all* $n \in \mathbb{N}_0$ *or*
- $c_{n+1,n} \neq 0$ *for all* $n \in \mathbb{N}_0$.

let $(\lambda_n)_{n\in\mathbb{N}_0}$ *be an* $\mathcal{R}^*$*-valued sequence, and let* $\rho_0 \in \mathcal{R}^*$.

- *For all* $n \in \mathbb{N}$*, let*

  $$\left|\left|\lambda_{n-1}a_{n-1,n}b_n^{-1}\lambda_n^{-1}\right|\right| + \left|\left|\lambda_{n+1}c_{n+1,n}b_n^{-1}\lambda_n^{-1}\right|\right| \leq 1. \tag{3.3.4}$$

  Then $\operatorname{gcf}(\mathbf{H})$ *converges, and for the speed of convergence,* (3.3.1) *holds with* $g_{N+1} = 1$ *and* $u_{n,n+1} = \left|\left|\lambda_{n+1}c_{n+1,n}b_n^{-1}\lambda_n^{-1}\right|\right|$.

- *If we have*

  $$\begin{aligned} \left|\left|\lambda_n c_{n,n-1}b_{n-1}^{-1}\lambda_{n-1}^{-1}\right|\right| + \left|\left|\lambda_n a_{n,n+1}b_{n+1}^{-1}\lambda_{n+1}^{-1}\right|\right| &\leq 1 \text{ for all } n \geq 2 \quad \text{and} \qquad (3.3.5)\\ \left|\left|\lambda_1 c_{10}\rho_0^{-1}\right|\right| + \left|\left|\lambda_1 a_{12}b_2^{-1}\lambda_2^{-1}\right|\right| &\leq 1, \qquad (3.3.6) \end{aligned}$$

  $\operatorname{gcf}(\mathbf{H})$ *converges, and for the speed of convergence,* (3.3.1) *holds with* $g_{N+1} = 1$ *and* $u_{n,n+1} = \left|\left|\lambda_n a_{n,n+1}b_{n+1}^{-1}\lambda_{n+1}^{-1}\right|\right|$.

For practical issues, we replace B_N by an easy-to-handle term.

Theorem 3.3.5. *Let* $\mathbf{H}$ *be tridiagonal, let* $b_n^{-1} \in \mathcal{R}^*$ *for all* $n \in \mathbb{N}$ *and let*

- $a_{n,n+1} \neq 0$ *for all* $n \in \mathbb{N}_0$ *or*
- $c_{n+1,n} \neq 0$ *for all* $n \in \mathbb{N}_0$.

Furthermore, let $(\lambda_n)_{n\in\mathbb{N}_0}$ *be an* $\mathcal{R}^*$*-valued sequence, let* $\rho_0 \in \mathcal{R}^*$*, let* $(\alpha_n)_{n\in\mathbb{N}}$ *be defined by* $\alpha_1 = \left|\left|\lambda_0 a_{01}b_1^{-1}\lambda_1^{-1}\right|\right| \cdot \left|\left|\lambda_1 c_{10}\rho_0^{-1}\right|\right|$ *and*

$$\alpha_n = \left|\left|\lambda_{n-1}a_{n-1,n}b_n^{-1}\lambda_n^{-1}\right|\right| \cdot \left|\left|\lambda_n c_{n,n-1}b_{n-1}^{-1}\lambda_{n-1}^{-1}\right|\right|, \qquad n \geq 2,$$

and let $q_1, q_2, \ldots > 1$ *be numbers with*

$$\alpha_n \leq \frac{q_n - 1}{q_{n-1}q_n}, \qquad n \geq 2. \tag{3.3.7}$$

Then $\operatorname{gcf}(\mathbf{H})$ *converges, and for all* $N \in \mathbb{N}_0$*, we have*

$$\left|\left|\lambda_0\left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right)\rho_0^{-1}\right|\right| \leq \frac{\alpha_1\alpha_2\ldots\alpha_{N+1}q_Nq_{N+1}}{C_N(q_NC_N - C_{N-1})(q_{N+1}-1)}, \tag{3.3.8}$$

where C_n *is defined by* $C_0 = C_1 = 1$ *and* $C_n = C_{n-1} - \frac{q_n-1}{q_{n-1}q_n}C_{n-2}$.

Proof. Note that $\alpha_n = u_{n-1,n}u_{n,n-1}$ for $\mathbf{U} = \hat{\mathbf{P}}$ or $\mathbf{U} = \hat{\mathbf{P}}^T$.

First, we prove that in (3.3.1), we can replace B_N by C_N, that is $B_N \geq C_N$. B_n was defined by $B_0 = B_1 = 1$ and $B_n = B_{n-1} - \alpha_n B_{n-2}$ for $n \geq 2$. Hence, we have $q_1B_1 - B_0 = q_1 - 1 = q_1C_1 - C_0$, and by induction, it follows that

$$\begin{aligned} q_nB_n - B_{n-1} &= (q_n-1)B_{n-1} - q_n\alpha_nB_{n-2} \geq (q_n-1)B_{n-1} - \frac{q_n-1}{q_{n-1}}B_{n-2} \\ &= \frac{q_n-1}{q_{n-1}}(q_{n-1}B_{n-1} - B_{n-2}) \geq \frac{q_n-1}{q_{n-1}}(q_{n-1}C_{n-1} - C_{n-2}) \\ &= (q_n-1)C_{n-1} - q_n\frac{q_n-1}{q_{n-1}q_n}C_{n-2} = q_nC_n - C_{n-1}. \end{aligned}$$

From this result, we derive $B_n - C_n \geq \frac{1}{q_n}(B_{n-1} - C_{n-1})$, and due to $B_0 - C_0 \geq 0$, $B_n \geq C_n$ for all $n \in \mathbb{N}_0$.

To complete the proof we show that a Pringsheim-Lyapunov function is given by

$$g_0 = 1, \qquad g_n = \frac{q_nu_{n,n-1}\dots u_{10}}{q_nC_n - C_{n-1}}, \qquad n \geq 1.$$

For $n = 1$, we have

$$\begin{aligned} u_{10}g_0 - g_1 + u_{12}g_2 &= u_{10}\left(1 - \frac{q_1}{q_1-1} + \frac{q_2u_{12}u_{21}}{q_2\left(1-\frac{q_2-1}{q_1q_2}\right)-1}\right) \\ &\leq u_{10}\left(1 - \frac{q_1}{q_1-1} + \frac{\frac{q_2-1}{q_1}}{q_2-1-\frac{q_2-1}{q_1}}\right) \\ &= u_{10}\left(1 - \frac{q_1}{q_1-1} + \frac{1}{q_1-1}\right) = 0. \end{aligned}$$

For $n \geq 2$, we have

$$\begin{aligned} & u_{n,n-1}g_{n-1} - g_n + u_{n,n+1}g_{n+1} \\ = \; & u_{n,n-1}\dots u_{10}\left(\frac{q_{n-1}}{q_{n-1}C_{n-1}-C_{n-2}} - \frac{q_n}{q_nC_n-C_{n-1}} + \frac{q_{n+1}u_{n,n+1}u_{n+1,n}}{q_{n+1}C_{n+1}-C_n}\right) \\ \leq \; & u_{n,n-1}\dots u_{10}\left(\frac{q_{n-1}}{q_{n-1}C_{n-1}-C_{n-2}} - \frac{q_n}{q_nC_n-C_{n-1}} + \frac{\frac{q_{n+1}-1}{q_n}}{(q_{n+1}-1)C_n - \frac{q_{n+1}-1}{q_n}C_{n-1}}\right) \\ = \; & u_{n,n-1}\dots u_{10}\left(\frac{q_{n-1}}{q_{n-1}C_{n-1}-C_{n-2}} - \frac{q_n}{q_nC_n-C_{n-1}} + \frac{1}{q_nC_n-C_{n-1}}\right) \\ = \; & u_{n,n-1}\dots u_{10}\left(\frac{q_{n-1}}{q_{n-1}C_{n-1}-C_{n-2}} - \frac{q_n-1}{q_nC_n-C_{n-1}}\right) \\ = \; & u_{n,n-1}\dots u_{10}\left(\frac{q_{n-1}}{q_{n-1}C_{n-1}-C_{n-2}} - \frac{q_n-1}{(q_n-1)C_{n-1}-\frac{q_n-1}{q_{n-1}}C_{n-2}}\right) \\ = \; & u_{n,n-1}\dots u_{10}\left(\frac{q_{n-1}}{q_{n-1}C_{n-1}-C_{n-2}} - \frac{q_{n-1}}{q_{n-1}C_{n-1}-C_{n-2}}\right) = 0. \end{aligned}$$

□

The easiest choice is $q_n = 2$, yielding $\frac{q_n - 1}{q_{n-1} q_n} = \frac{1}{4}$, $C_n = \frac{n+1}{2^n}$ and $C_N(q_N C_N - C_{N-1}) = \frac{N+1}{2^{2N-1}}$.

Corollary 3.3.1. *Let* $\mathbf{H}$ *be tridiagonal, let* $b_n^{-1} \in \mathcal{R}^*$ *for all* $n \in \mathbb{N}$ *and let*

- $a_{n,n+1} \neq 0$ *for all* $n \in \mathbb{N}_0$ *or*
- $c_{n+1,n} \neq 0$ *for all* $n \in \mathbb{N}_0$.

Furthermore, let $(\lambda_n)_{n\in\mathbb{N}_0}$ *and* $(\rho_n)_{n\in\mathbb{N}_0}$ *be* $\mathcal{R}^*$*-valued sequences, let* $(\alpha_n)_{n\in\mathbb{N}}$ *be defined as in Theorem* 3.3.5*, and let* $\alpha_n \leq \frac{1}{4}$ *for* $n \geq 2$*. Then* $\operatorname{gcf}(\mathbf{H})$ *converges with*

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq \frac{2^{2N+1}}{N+1} \alpha_1 \alpha_2 \dots \alpha_{N+1} \leq \frac{2\alpha_1}{N+1}.$$

Remark 3.3.1. In $\mathbb{C}$, due to commutativity we can assume $c_{n+1,n} = 1$, and the equivalence transformations can be used for guaranteeing $b_n = 1$. Then we obtain convergence whenever $|a_{n,n+1}| \leq \frac{1}{4}$ for $n \geq 2$. Hence, Corollary 3.3.1 is a *Worpitzky–type* criterion, see section 1.2. Note that α_n is invariant with respect to scalar equivalence transformations. Hence, it is not possible to derive the general statement of 3.3.5 from Corollary 3.3.1 by means of equivalence transformations.

Remark 3.3.2. In all of our Pringsheim–type or Worpitzky–type criteria, we used terms containing $a_{mn} b_n^{-1}$ and $c_{mn} b_n^{-1}$. There may be situations in which the terms $b_m^{-1} a_{mn}$ and $b_m^{-1} c_{mn}$ are easier to handle. If this is the case, we recommend replacing $\lambda_n = \rho_n b_n^{-1}$.

3.4 Literature review

As mentioned in section 2.4, there is no literature dealing with continued fractions in the generality of Definition 2.3.1. Hence, there is no Pringsheim-type convergence criterion which can be compared with Theorem 3.2.3 or Theorem 3.2.5.

In the case of upper Hessenberg matrices $\mathbf{H}$, the definition of $\operatorname{gcf}(\mathbf{H})$ and the Pringsheim-type criterion simplifies (see section 3.3). Nevertheless, to the knowledge of the author, all results for this special case are new. However, if $c_{n+1,n} \in \mathcal{R}^*$ for all $n \in \mathbb{N}_0$, the approximants $K_0^{(N)}$ can be written as $K_0^{(N)} = A_N B_N^{-1}$, where the sequences (A_n) and (B_n) satisfy a simple recurrence relation. For gcfs generated in such a manner, there is some literature. We will come back to this topic in chapter 7, where we will give alternative proofs for some results. Then, we will compare our statements for this special case with results found in the literature.

For tridiagonal matrices $\mathbf{H}$, some results without unnecessary invertibility assumptions can be found in the literature. In general, tridiagonal matrices $\mathbf{H}$ generate non-generalized continued fractions. As pointed out in section 1.2, first theoretical results were found in the 1960s and 1970s. The first criterion which can be interpreted as a generalization of Pringsheim's criterion without the assumption $c_{n+1,n} \in \mathcal{R}^*$ is due to Denk and Riederle [DR82]. Up to notation, they used a condition quite similar to (3.3.7) in Theorem 3.3.5. They proved that (3.3.7) guarantees that $K_0^{(N)}$ is well-defined for all $N \in \mathbb{N}$ (that is, $K_n^{(N)} \in \mathcal{R}^*$ for all $n = 1, \dots, N$), and found an upper bound for $\left|\left|\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right|\right|$. Unfortunately, this upper bound was slightly larger

than that in (3.3.8), and (3.3.7) did not guarantee convergence of the upper bound to 0. Hence, they needed an additional condition for stating convergence, they explicitly assumed that the upper bound converges to 0. In the special case of $q_n = \frac{1}{2}$, we are able to state convergence for $\alpha_n \leq \frac{1}{4}$ for $n \geq 2$ (see Corollary 3.3.1). Denk and Riederle assumed

$$4^N \prod_{n=1}^{N} \alpha_n \to 0$$

additionally, which obviously is restrictive: Corollary 3.3.1 guarantees convergence even in case $\alpha_n = \frac{1}{4}$ for all $n \geq 2$ with $\left|\left|\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right|\right| \leq \frac{2\alpha_1}{N+1}$.

Another contribution to the topic of Pringsheim-type criteria for (non-generalized) continued fractions on Banach algebras is due to Schelling [Sch96]. With our notation, he defined

$$\begin{aligned} \tilde{\alpha}_n &= \left|\left|\lambda_{n-1} a_{n-1,n} b_n^{-1} \lambda_n^{-1}\right|\right| \cdot ||\lambda_n c_{n,n-1}|| \cdot \left|\left|b_{n-1}^{-1} \lambda_{n-1}^{-1}\right|\right|, \qquad n \geq 2 \qquad \text{or} \\ \tilde{\alpha}_n &= \left|\left|\rho_{n-1} b_{n-1}^{-1}\right|\right| \cdot \left|\left|a_{n-1,n} \rho_n^{-1}\right|\right| \cdot \left|\left|\rho_n b_n^{-1} c_{n,n-1} \rho_{n-1}^{-1}\right|\right|, \qquad n \geq 2 \end{aligned}$$

and proved convergence for $\tilde{\alpha}_n \leq (1-\epsilon)\frac{q_n - 1}{q_{n-1} q_n}$ with some constant $\epsilon > 0$. Submultiplicativity guarantees $\alpha_n \leq \tilde{\alpha}_n$, and in Corollary 3.3.1 we proved convergence for $\epsilon = 0$, and hence, our results are much stronger. In fact, the ϵ-condition guarantees that the additional condition in [DR82] is met. In the second part of [Sch96], Schelling proved some results for the special case where $c_{n+1,n} \in \mathcal{R}^*$. Since tridiagonal matrices are a special case of upper Hessenberg matrices, we will come back to this topic in chapter 7.

Both in [DR82] and [Sch96], the proof for convergence was based on contractive mappings whereas our proofs of Pringsheim-type convergence criteria is completely different. For tridiagonal $\mathbf{H}$, this technique was developed in [Bau10], where the convergence criteria as given in Theorem 3.3.4 and Theorem 3.3.5 were already stated. In [Bau15], the speed of convergence statements were slightly improved and correspond exactly to those given in Theorem 3.3.5 and Corollary 3.3.1.

Finally, we remark that the equivalence transformations used here are only a slight generalization to those in [DR82] and [Sch96] for non-generalized continued fractions in Banach algebras.

An application of (non-generalized) matrix-valued continued fractions (without the assumption $c_{n+1,n} \in \mathcal{R}^*$) is given by Bobryk [Bob07]: The author truncated an infinite system of difference-differential equations, and obtained a finite one (which can be solved numerically). Since he was able to write the error in terms of continued fractions, he used the results from [DR82] for proving convergence of his method and for obtaining an error bound. As pointed out in [Bau15], (3.3.8) slightly improves this error bound.

Chapter 4

GCFs and infinite systems of equations

In section 2.1, we pointed out that the system $\psi\mathbf{P} = \psi$ for computing the invariant measure for an irreducible stochastic matrix $\mathbf{P}$ is solved by means of continued fractions in case of tridiagonality.

To be precise, in case of a (block-)tridiagonal matrix, we stated that

$$\psi_n^{(N)} = \psi_{n-1}^{(N)} p_{n-1,n} \left(K_n^{(N)}\right)^{-1}.$$

Using S-series and the notation $R_{k,n,m}^{(N)}$ from Definition 2.3.1 (with $-a_{n-1,n} = p_{n-1,n}$), this relationship can be rewritten as

$$\psi_n^{(N)} = \psi_{n-1}^{(N)} S(\mathbf{P}, n-1, n, \mathbb{N}_{n,N}) = \psi_{n-1}^{(N)} R_{n-1,n,n}^{(N)},$$

and in case of tridiagonality, (2.3.5) yields

$$\psi_n^{(N)} = \psi_0 S(\mathbf{P}, 0, n, \mathbb{N}_{1,N}) = \psi_0 R_{0,1,n}^{(N)}.$$

For Markov chains, we will see that this relationship holds for arbitrarily structured transition probability matrices.

Invariant measures fulfill $\psi\mathbf{P} = \psi$, or equivalently, $\psi\mathbf{H} = 0$. In many practical situations, it is more common to write homogeneous systems of linear equations as $\mathbf{H}x = 0$, and in the context of Markov chains, the system which characterizes absorption probabilities has this form. A natural approach for obtaining a gcf-related solution for this transposed problem is setting

$$x_n^{(N)} = L_{n,1,0}^{(N)} x_0.$$

In this chapter, we will proceed as follows:

- In section 4.1, we will prove that $\left(x_n^{(N)}\right)_{n=0}^{N}$ fulfills an appropriately truncated system of linear equations in case that the inverses in the definition of $L_{n,1,0}^{(N)}$ exist. Actually, the procedure can be interpreted as a variant of Gaussian elimination, as we will state in section 4.10.

- If we have solutions $\left(x_n^{(N)}\right)_{n=0}^{N}$ for truncated systems, the question arises whether or not the limit is a solution for the untruncated system. This is not true in general. Nevertheless, we will find criteria which guarantee that $x_n = \lim_{N\to\infty} x_n^{(N)}$ exists and that this limit fulfills $\mathbf{H}x = 0$. The first such criteria will be given in terms of S-series (see section 4.4), after that, we will derive Pringsheim-type conditions (see section 4.5).

- In the context of computing absorption probabilities for Markov chains, we have fixed $x_0 = 1$. Actually, in such a situation, $\mathbf{H}x = 0$ is an inhomogeneous system. In section 4.6, we will briefly discuss how to solve general inhomogeneous systems by means of gcfs.

- Naturally, all these results can be applied to systems of the form $\psi\mathbf{H} = 0$ by transposition, we will summarize the results in section 4.7.

- In general, the system $\mathbf{H}x = 0$ will have many solutions, even if $x_0 = 1$ is fixed. For Markov chains, absorption probabilities are characterized as the minimal positive solution of this system. Since our gcf-based approach for the general problem is inspired by Markov chains, we may hope for the property of minimality to hold in general. It turns out that the S-series (and hence, the Pringsheim-type) criteria guarantee this minimality (see section 4.8).

- In the context of Markov chains, we often deal with block matrices $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$, that is, the p_{ij} are matrices, see section 1.3. In many of these applications, we cannot apply our results directly since we have $p_{ij} \in \mathbb{R}^{d_i\times d_j}$ (the levels do not have the same size). Therefore, we cannot directly use results in which we assume that $p_{ij} \in \mathcal{R}$ for all $i, j \in \mathbb{N}_0$ with some Banach algebra $\mathcal{R}$. In section 4.9, we will discuss how to deal with this problem.

4.1 The system $\mathbf{H}x = 0$

With $\mathbf{H}$ as defined above, the system $\mathbf{H}x = 0$ can be written as

$$0 = b_0x_0 + \sum_{m=1}^{\infty} a_{0m}x_m, \tag{4.1.1}$$

$$\sum_{m=0}^{n-1} c_{nm}x_m = b_nx_n + \sum_{m=n+1}^{\infty} a_{nm}x_m, \qquad n \in \mathbb{N}. \tag{4.1.2}$$

In case of a tridiagonal matrix $\mathbf{H}$, (4.1.2) is a second-order difference equation, and (4.1.1) is some initial condition in terms of x_0 and x_1. For many structured matrices $\mathbf{H}$, such an interpretation holds. Therefore, we will refer to (4.1.1) as *initial condition* for (4.1.2) in any case.

The system $\mathbf{H}x = 0$ may have infinitely many solutions, and in many situations there is no way of finding explicit representations of any solution. Naturally, numerical methods can only be applied directly to finite systems of linear equations. Hence, an idea might be solving a

truncated system, that is

$$0 = b_0 x_0 + \sum_{m=1}^{N} a_{0m} x_m, \tag{4.1.3}$$

$$\sum_{m=0}^{n-1} c_{nm} x_m = b_n x_n + \sum_{m=n+1}^{N} a_{nm} x_m, \qquad n = 1, \ldots, N \tag{4.1.4}$$

for large $N \in \mathbb{N}$. Using the notation $\mathbf{H}_{n,N}$ for submatrices again, we can write this system as $\mathbf{H}_{0,N} x = 0$. In this context, we have to state that the existence of non-trivial solutions of $\mathbf{H}x = 0$ does not imply the existence of non-trivial solutions of the truncated system $\mathbf{H}_{0,N} x = 0$.

Example 4.1.1. Let $\mathcal{R} = \mathbb{C}$, let $\mathbf{H}$ be tridiagonal, and let $a_{n,n+1} \neq 0, b_n \neq 0, c_{n+1,n} \neq 0$ for all $n \in \mathbb{N}_0$. Then, (4.1.2) is a second-order difference equation with a two-dimensional space of solutions. The initial condition (4.1.1) characterizes a one-dimensional subspace. Hence, there is a non-trivial solution (which is unique up to a constant factor). On the other hand, $\mathbf{H}_{0,N}$ is non-singular in general, and hence, $\mathbf{H}_{0,N} x = 0$ yields $x = 0$.

Hence, it makes no sense solving $\mathbf{H}_{0,N} x = 0$ completely. Instead, we focus on (4.1.4) and ignore the initial condition (4.1.3) of the truncated system. (4.1.4) can be interpreted as solving (4.1.2) with the additional constraints $x_{N+1} = x_{N+2} = \ldots = 0$. Hence, an approach for solving $\mathbf{H}x = 0$ is:

- Find a solution $x^{(N)}$ for (4.1.4), that is, a solution $x^{(N)}$ for (4.1.2) subject to $x^{(N)}_{N+1} = x^{(N)}_{N+2} = \ldots = 0$.
- Take $N \to \infty$.

Some questions and tasks arise:

- Find conditions for (4.1.4) to have a nontrivial solution $x^{(N)}$.
- Define $x^{(N)}_n = 0$ for $n > N$. Does $x_n := \lim\limits_{N\to\infty} x^{(N)}_n$ converge for all $n \in \mathbb{N}_0$?
- If $x = (x_n)$ does converge: Is x nontrivial and does x satisfy (4.1.2)?
- Does x also satisfy the initial condition (4.1.1)?

Our goal is to find answers in terms of continued fractions and their generalizations. We begin with giving a gcf-based solution for the system (4.1.2) subject to $x_{N+1} = x_{N+2} = \ldots = 0$, that is, a solution for (4.1.4). This is done in Theorem 4.1.1 below. We have to admit that for finite systems of the form (4.1.4), the gcf-based solution can be obtained by Gaussian elimination, see section 4.10 for more details. In particular, the induction step in the proof of Theorem 4.1.1 below can be interpreted as a reduction step in the sense of Gaussian elimination. In what follows, $L^{(N)}_{m,n,k}$ shall be defined as in Definition 2.3.1, and we prove an auxiliary result first.

Lemma 4.1.1. *For $0 \le k < n \le m \le N$, we have*

$$L^{(N)}_{m,n,k} = L^{(N)}_{m,m,k} + \sum_{\ell=n}^{m-1} L^{(N)}_{m,m,\ell} L^{(N)}_{\ell,n,k} \tag{4.1.5}$$

Proof. The proof is performed by induction with respect to $m-n$, where for $m-n=0$ the statement is trivial. Now let it be true for differences $m-n-1$. Then we obtain

$$\begin{aligned}
L^{(N)}_{m,n,k} &\overset{(2.3.3)}{=} L^{(N)}_{m,n+1,k} + L^{(N)}_{m,n+1,n} L^{(N)}_{n,n,k} \\
&= L^{(N)}_{m,m,k} + \sum_{\ell=n+1}^{m-1} L^{(N)}_{m,m,\ell} L^{(N)}_{\ell,n+1,k} + \left(L^{(N)}_{m,m,n} + \sum_{\ell=n+1}^{m-1} L^{(N)}_{m,m,\ell} L^{(N)}_{\ell,n+1,n} \right) L^{(N)}_{n,n,k} \\
&= L^{(N)}_{m,m,k} + L^{(N)}_{m,m,n} L^{(N)}_{n,n,k} + \sum_{\ell=n+1}^{m-1} L^{(N)}_{m,m,\ell} \left(L^{(N)}_{\ell,n+1,k} + L^{(N)}_{\ell,n+1,n} L^{(N)}_{n,n,k} \right) \\
&\overset{(2.3.3)}{=} L^{(N)}_{m,m,k} + L^{(N)}_{m,m,n} L^{(N)}_{n,n,k} + \sum_{\ell=n+1}^{m-1} L^{(N)}_{m,m,\ell} L^{(N)}_{\ell,n,k} \\
&= L^{(N)}_{m,m,k} + \sum_{\ell=n}^{m-1} L^{(N)}_{m,m,\ell} L^{(N)}_{\ell,n,k}.
\end{aligned}$$

□

Theorem 4.1.1. *Let $N \in \mathbb{N}$, and let $L^{(N)}_{n,1,0}$ be well-defined for all $n = 1, \dots, N$ and define $x_n = L^{(N)}_{n,1,0} x_0$ for $n \ge 1$ with some $x_0 \in \mathcal{R}$. Then $(x_n)_{n=0}^N$ solves* (4.1.4)*, and $(x_n)_{n\in\mathbb{N}_0}$ solves* (4.1.2)*.*

Proof. Due to $L^{(N)}_{n,1,0} = 0$ for $n > N$, it suffices to prove that $(x_n)_{n=0}^N$ solves (4.1.4), and obviously, we may assume $x_0 = I$ since a homogeneous multiplication of (x_n) from the right does not change the property of (x_n) solving the system of equations.

For $x_0 = I$, the statement is proved by induction with respect to N, and as we have already mentioned, the induction step is quite similar to a reduction step of Gaussian elimination, see section 4.10 for more details.

For $N = 1$, we have $L^{(1)}_{1,1,0} = b_1^{-1} c_{10}$, and the only equation

$$c_{10} x_0 = b_1 x_1$$

is obviously fulfilled by $x_0 = I$, $x_1 = L^{(1)}_{1,1,0} = b_1^{-1} c_{10}$.

Now let the statement be true for all systems truncated after row and column N, that is we have

$$\tilde{c}_{n0} + \sum_{m=1}^{n-1} \tilde{c}_{nm} \tilde{L}^{(N)}_{m,1,0} = \tilde{b}_n \tilde{L}^{(N)}_{n,1,0} + \sum_{m=n+1}^{N} \tilde{a}_{nm} \tilde{L}^{(N)}_{m,1,0}, \qquad n = 1, \dots, N$$

for any choice of $\tilde{b}_n$, $\tilde{a}_{nm}$, $\tilde{c}_{nm}$, where $\tilde{L}^{(N)}_{n,1,0}$ is defined by means of (2.3.1), (2.3.2) and (2.3.3) using the coefficients $\tilde{b}_n, \ldots$ instead of $b_n, \ldots$.

Consider an arbitrary system with the usual coefficients b_n, a_{nm}, c_{nm} truncated at row and column $N+1$. Define

$$\begin{aligned}\tilde{b}_n &= b_n + a_{n,N+1} b_{N+1}^{-1} c_{N+1,n},\\ \tilde{a}_{nm} &= a_{nm} + a_{n,N+1} b_{N+1}^{-1} c_{N+1,m},\\ \tilde{c}_{nm} &= c_{nm} - a_{n,N+1} b_{N+1}^{-1} c_{N+1,m}.\end{aligned}$$

First, we prove that $\tilde{K}^{(N)}_n = K^{(N+1)}_n$ and $\tilde{L}^{(N)}_{m,n,k} = L^{(N+1)}_{m,n,k}$ for $0 \le k < n \le m \le N$ by (backward) induction with respect to n, using $b^{-1}_{N+1} c_{N+1,k} = L^{(N+1)}_{N+1,N+1,k}$. For $n = N$, we have

$$\begin{aligned}\tilde{K}^{(N)}_N &= \tilde{b}_N = b_N + a_{N,N+1} b^{-1}_{N+1} c_{N+1,N} = b_N + a_{N,N+1} L^{(N+1)}_{N+1,N+1,N} = K^{(N+1)}_N,\\ \tilde{L}^{(N)}_{N,N,k} &= \left(\tilde{K}^{(N)}_n\right)^{-1} \tilde{c}_{Nk} = \left(K^{(N+1)}_N\right)^{-1} \left(c_{Nk} - a_{N,N+1} b^{-1}_{N+1} c_{N+1,k}\right)\\ &= \left(K^{(N+1)}_N\right)^{-1} \left(c_{Nk} - a_{N,N+1} L^{(N+1)}_{N+1,N+1,k}\right) = L^{(N+1)}_{N,N,k}.\end{aligned}$$

Now, let $n < N$ and let the statement be true for $n+1$. Then we have

$$\begin{aligned}\tilde{K}^{(N)}_n &= \tilde{b}_n + \sum_{m=n+1}^{N} \tilde{a}_{nm} \tilde{L}^{(N)}_{m,n+1,n}\\ &= b_n + a_{n,N+1} b^{-1}_{N+1} c_{N+1,n} + \sum_{m=n+1}^{N} \left(a_{nm} + a_{n,N+1} b^{-1}_{N+1} c_{N+1,m}\right) L^{(N+1)}_{m,n+1,n}\\ &= b_n + \sum_{m=n+1}^{N} a_{nm} L^{(N+1)}_{m,n+1,n} + a_{n,N+1} \left(L^{(N+1)}_{N+1,N+1,n} + \sum_{m=n+1}^{N} L^{(N+1)}_{N+1,N+1,m} L^{(N+1)}_{m,n+1,n}\right)\\ &\overset{(4.1.5)}{=} b_n + \sum_{m=n+1}^{N} a_{nm} L^{(N+1)}_{m,n+1,n} + a_{n,N+1} L^{(N+1)}_{N+1,n+1,n} = K^{(N+1)}_n,\\ \tilde{L}^{(N)}_{n,n,k} &= \left(\tilde{K}^{(N)}_n\right)^{-1} \left(\tilde{c}_{nk} - \sum_{m=n+1}^{N} \tilde{a}_{nm} \tilde{L}^{(N)}_{m,n+1,k}\right)\\ &= \left(K^{(N+1)}_n\right)^{-1} \left(c_{nk} - a_{n,N+1} b^{-1}_{N+1} c_{N+1,k} - \sum_{m=n+1}^{N} \left(a_{nm} + a_{n,N+1} b^{-1}_{N+1} c_{N+1,m}\right) L^{(N+1)}_{m,n+1,k}\right)\\ &= \left(K^{(N+1)}_n\right)^{-1} \left(c_{nk} - \sum_{m=n+1}^{N} a_{nm} L^{(N+1)}_{m,n+1,k} - a_{n,N+1} \right.\\ &\qquad \left. \cdot \left(L^{(N+1)}_{N+1,N+1,k} + \sum_{m=n+1}^{N} L^{(N+1)}_{N+1,N+1,m} L^{(N+1)}_{m,n+1,k}\right)\right)\\ &\overset{(4.1.5)}{=} \left(K^{(N+1)}_n\right)^{-1} \left(c_{nk} - \sum_{m=n+1}^{N} a_{nm} L^{(N+1)}_{m,n+1,k} - a_{n,N+1} L^{(N+1)}_{N+1,n+1,k}\right) = L^{(N+1)}_{n,n,k},\\ \tilde{L}^{(N)}_{m,n,k} &= \tilde{L}^{(N)}_{m,n+1,k} + \tilde{L}^{(N)}_{m,n+1,n} \tilde{L}^{(N)}_{n,n,k}\\ &= L^{(N+1)}_{m,n+1,k} + L^{(N+1)}_{m,n+1,n} L^{(N+1)}_{n,n,k} = L^{(N+1)}_{m,n+1,k}.\end{aligned}$$

With these preparations, we are able to prove that the truncated system of linear equations

is fulfilled. For $n = 1, \dots, N$, we have

$$\tilde{c}_{n0} + \sum_{m=1}^{n-1} \tilde{c}_{nm} \tilde{L}^{(N)}_{m,1,0} = \tilde{b}_n \tilde{L}^{(N)}_{n,1,0} + \sum_{m=n+1}^{N} \tilde{a}_{nm} \tilde{L}^{(N)}_{m,1,0}$$

and thus

$$\begin{aligned}
& c_{n0} + \sum_{m=1}^{n-1} c_{nm} L^{(N+1)}_{m,1,0} \\
= \;& \tilde{c}_{n0} + a_{n,N+1} b^{-1}_{N+1} c_{N+1,0} + \sum_{m=1}^{n-1} \tilde{c}_{nm} L^{(N+1)}_{m,1,0} + \sum_{m=1}^{n-1} a_{n,N+1} b^{-1}_{N+1} c_{N+1,m} L^{(N+1)}_{m,1,0} \\
= \;& \tilde{c}_{n0} + \sum_{m=1}^{n-1} \tilde{c}_{nm} \tilde{L}^{(N)}_{m,1,0} + a_{n,N+1} \left(L^{(N+1)}_{N+1,N+1,0} + \sum_{m=1}^{n-1} L^{(N+1)}_{N+1,N+1,m} L^{(N+1)}_{m,1,0} \right) \\
= \;& \tilde{b}_n \tilde{L}^{(N)}_{n,1,0} + \sum_{m=n+1}^{N} \tilde{a}_{nm} \tilde{L}^{(N)}_{m,1,0} + a_{n,N+1} \left(L^{(N+1)}_{N+1,N+1,0} + \sum_{m=1}^{n-1} L^{(N+1)}_{N+1,N+1,m} L^{(N+1)}_{m,1,0} \right) \\
= \;& \left(b_n + a_{n,N+1} b^{-1}_{N+1} c_{N+1,n} \right) L^{(N+1)}_{n,1,0} + \sum_{m=n+1}^{N} \left(a_{nm} + a_{n,N+1} b^{-1}_{N+1} c_{N+1,m} \right) L^{(N+1)}_{m,1,0} \\
& \quad + a_{n,N+1} \left(L^{(N+1)}_{N+1,N+1,0} + \sum_{m=1}^{n-1} L^{(N+1)}_{N+1,N+1,m} L^{(N+1)}_{m,1,0} \right) \\
= \;& b_n L^{(N+1)}_{n,1,0} + \sum_{m=n+1}^{N} a_{nm} L^{(N+1)}_{m,1,0} + a_{n,N+1} \left(L^{(N+1)}_{N+1,N+1,0} + \sum_{m=1}^{N} L^{(N+1)}_{N+1,N+1,m} L^{(N+1)}_{m,1,0} \right) \\
\overset{(4.1.5)}{=} \;& b_n L^{(N+1)}_{n,1,0} + \sum_{m=n+1}^{N} a_{nm} L^{(N+1)}_{m,1,0} + a_{n,N+1} L^{(N+1)}_{N+1,1,0} \\
= \;& b_n L^{(N+1)}_{n,1,0} + \sum_{m=n+1}^{N+1} a_{nm} L^{(N+1)}_{m,1,0},
\end{aligned}$$

and for $n = N + 1$, from (4.1.5) and $L^{(N+1)}_{N+1,N+1,m} = b^{-1}_{N+1} c_{N+1,m}$, it directly follows that

$$b^{-1}_{N+1} c_{N+1,0} + \sum_{m=1}^{N} b^{-1}_{N+1} c_{N+1,m} L^{(N+1)}_{m,1,0} = L^{(N+1)}_{N+1,1,0},$$

that is

$$c_{N+1,0} + \sum_{m=1}^{N} c_{N+1,m} L^{(N+1)}_{m,1,0} = b_{N+1} L^{(N+1)}_{N+1,1,0}.$$

□

It is quite natural to conjecture that taking $N \to \infty$ yields a solution $x_n = L_{n,1,0} x_0$ for (4.1.2). Hence, we introduce

Definition 4.1.1. Let $\mathbf{H}$, $L^{(N)}_{m,n,k}$ and $L_{m,n,k}$ be as defined in Definition 2.3.1. For arbitrary $w_0 \in \mathcal{R}$, we write

$$w^{(N)}_n(w_0, \mathbf{H}) := \begin{cases} w_0, & n = 0, \\ L^{(N)}_{n,1,0} w_0, & n = 1, \dots, N, \\ 0, & n > N. \end{cases} \qquad \text{and} \qquad w_n(w_0, \mathbf{H}) := L_{n,1,0} w_0, \; n \in \mathbb{N}_0.$$

We write $w^{(N)}(w_0, \mathbf{H}) = \left(w_n^{(N)}\right)_{n\in\mathbb{N}_0}$ and $w(w_0, \mathbf{H}) = (w_n)_{n\in\mathbb{N}_0}$ for the corresponding (infinite) column vectors.

Provided that $L_{n,1,0}^{(N)}$ is well-defined for all $n = 1, \ldots, N$, Theorem 4.1.1 guarantees that $w^{(N)}(w_0, \mathbf{H})$ solves (4.1.2) for any choice of w_0. With these definitions, we restate some of the questions and tasks:

- Is $L_{n,1,0}^{(N)}$, and thus, $w_n^{(N)}(w_0, \mathbf{H})$ well-defined?
- Does $w(w_0, \mathbf{H})$ converge?
- Is $w(w_0, \mathbf{H})$ a solution of (4.1.2)?
- Does $w(w_0, \mathbf{H})$ solve additionally (4.1.1)?

Before we will discuss these questions, we consider some special cases.

4.2 Special cases

4.2.1 Tridiagonal matrices: Second-order difference equations

Consider the tridiagonal matrix

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & & & \\ -c_{10} & b_1 & a_{12} & & \\ & -c_{21} & b_2 & a_{23} & \\ & & -c_{32} & b_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Here, (4.1.2) simplifies to the difference equation

$$c_{n,n-1}x_{n-1} = b_n x_n + a_{n,n+1}x_{n+1}, \qquad n \in \mathbb{N}, \tag{4.2.1}$$

and the initial condition (4.1.1) becomes

$$0 = b_0 x_0 + a_{01} x_1. \tag{4.2.2}$$

For tridiagonal matrices, the recursions for the $L_{m,n,k}^{(N)}$ simplify to

$$L_{n,n,n-1}^{(N)} = c_{n,n-1}\left(b_n + a_{n,n+1}L_{n+1,n+1,n}^{(N)}\right)^{-1}, \qquad n \in \mathbb{N} \tag{4.2.3}$$

$$L_{n,1,0}^{(N)} = \prod_{1}^{\ell=n} L_{\ell,\ell,\ell-1}^{(N)}, \qquad n \in \mathbb{N}. \tag{4.2.4}$$

(4.2.4) directly yields the recursion $w_n^{(N)} = L_{n,n,n-1}^{(N)} w_{n-1}^{(N)}$ and – in case of convergence – $w_n = L_{n,n,n-1} w_{n-1}$ for $n \in \mathbb{N}$.

It is clear that in case of convergence, (w_n) solves the difference equation (4.2.1). For second-order difference equations in $\mathbb{C}$, there is an algorithm for computing subdominant solutions, known as *Miller's algorithm*. In principle, this algorithm is based on backward computation. In section 8.1, we will demonstrate that Miller's algorithm can be interpreted as a variant of our gcf-based solution $w(w_0, \mathbf{H})$.

4.2.2 Band matrices: Higher order difference equations

Assume $c_{nm} = 0$ for $m \leq n-2$ and $a_{nm} = 0$ for $m \geq n+d+1$, that is

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & \dots & a_{0d} & & & \\ -c_{10} & b_1 & a_{12} & \dots & a_{1,d+1} & & \\ & -c_{21} & b_2 & a_{23} & \dots & a_{2,d+2} & \\ & & \ddots & \ddots & \ddots & \dots & \ddots \end{pmatrix}.$$

Here, (4.1.2) simplifies to

$$c_{n,n-1}x_{n-1} = b_n x_n + \sum_{m=n+1}^{n+d} a_{nm}x_m, \tag{4.2.5}$$

that is, a difference equation of order $d+1$. Again, for $\mathcal{R} = \mathbb{C}$, $w^{(N)}(w_0, \mathbf{H})$ may be obtained by Miller's backward computation (see section 8.1). Still, if $w = \lim_{N\to\infty} w^{(N)}$ converges, w will satisfy (4.2.5). The conjecture that w is a subdominant solution of (4.2.5) is not true anymore: Zahar [Zah77] found an example for a third-order difference equation with a subdominant solution, but Miller's backward algorithm (and hence, the gcf-based w) converges to a non-subdominant solution. We will briefly summarize this example in section 8.1.

Zahar pointed out that for higher-order difference equations, the convergence of the backward-computation algorithm can be characterized in terms of *dominant solutions* of the *adjoint system*. We will focus on adjoint systems in a slightly more general setting in chapter 8. However, in some situations w is a minimal or a subdominant solution. We will discuss these situations throughout the next chapters.

4.2.3 Upper Hessenberg matrices: Sum equations

For an upper Hessenberg matrix

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & a_{02} & a_{03} & \cdots \\ -c_{10} & b_1 & a_{12} & a_{13} & \cdots \\ & -c_{21} & b_2 & a_{23} & \cdots \\ & & -c_{32} & b_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

(4.1.2) simplifies to

$$c_{n,n-1}x_{n-1} = b_n x_n + \sum_{m=n+1}^{\infty} a_{nm}x_m, \qquad n \in \mathbb{N}. \tag{4.2.6}$$

We will refer to equations of this type as *sum equation*. As for difference equations, we may obtain $w^{(N)}(w_0, \mathbf{H})$ by backward computation in some situations (see section 8.1).

Theorem 4.2.1. *Let $c_{n,n-1} \in \mathcal{R}^*$ for all $n \in \mathbb{N}$ and let $w^{(N)}(w_0, \mathbf{H})$ be well-defined. Then $w^{(N)}(w_0, \mathbf{H})$ can be obtained by the following algorithm*

- *Set $x_{N+1} = x_{N+2} = \ldots = 0$ and $x_N = I$.*
- *Compute $x_{N-1}, \ldots, x_0$ by (4.2.6).*
- *Multiply all x_n by $x_0^{-1} w_0$.*

Due to infinitely many non-trivial summands in (4.2.6), even in the case of convergence of $w(w_0, \mathbf{H})$, there are situations in which $w(w_0, \mathbf{H})$ does not solve (4.2.6). We give an example in which there is no non-trivial solution at all.

Example 4.2.1. Consider the system $\mathbf{H}x = 0$ with

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 1 & \cdots \\ & 2 & 1 & 1 & \cdots \\ & & 2 & 1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix},$$

that is

$$-2x_{n-1} = \sum_{m=n}^{\infty} x_m, \qquad n \in \mathbb{N}, \tag{4.2.7}$$

Clearly, (4.2.7) has no non-trivial solution, since we have

$$-2x_{n-1} = \sum_{m=n}^{\infty} x_m = x_n + \sum_{m=n+1}^{\infty} x_m = x_n - 2x_n = -x_n,$$

and thus, any solution has to be of the form $x_n = 2^n x_0$, and on the other hand, the only solution of this form is $x = (0)$.

The truncated system is given by $-2x_{n-1} = \sum_{m=n}^{N} x_m$ for $n = 1, \ldots, N$, and for any choice of x_0, this system is obviously solved by $x_n = 2^n x_0$ for $n = 1, 2, \ldots, N-1$ and $x_N = -2^N x_0$. Obviously, this solution can be obtained by backward computation, that is, it coincides with the gcf-based solution $w^{(N)}(w_0, \mathbf{H})$.

4.3 Equivalence transformations for w

In section 3.1, we discussed equivalence transformations for gcfs. In this context, we considered

$$\tilde{b}_n = \lambda_n b_n \rho_n^{-1},\ \tilde{a}_{nm} = \lambda_n a_{nm} \rho_m^{-1},\ \tilde{c}_{mn} = \lambda_m c_{mn} \rho_n^{-1},$$

and proved $\tilde{K}_n^{(N)} = \lambda_n K_n^{(N)} \rho_n^{-1}$ and $\tilde{L}_{m,n,k}^{(N)} = \rho_m L_{m,n,k}^{(N)} \rho_k^{-1}$.

Hence, with $\tilde{w}_0 = \rho_0 w_0$, we have $\tilde{L}_{n,1,0}^{(N)} \tilde{w}_0 = \rho_n L_{n,1,0}^{(N)} w_0$. We directly obtain

Theorem 4.3.1. *Let*

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & a_{02} & a_{03} & \cdots \\ -c_{10} & b_1 & a_{12} & a_{13} & \cdots \\ -c_{20} & -c_{21} & b_2 & a_{23} & \cdots \\ -c_{30} & -c_{31} & -c_{32} & b_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad \textit{and} \quad \tilde{\mathbf{H}} = \begin{pmatrix} \tilde{b}_0 & \tilde{a}_{01} & \tilde{a}_{02} & \tilde{a}_{03} & \cdots \\ -\tilde{c}_{10} & \tilde{b}_1 & \tilde{a}_{12} & \tilde{a}_{13} & \cdots \\ -\tilde{c}_{20} & -\tilde{c}_{21} & \tilde{b}_2 & \tilde{a}_{23} & \cdots \\ -\tilde{c}_{30} & -\tilde{c}_{31} & -\tilde{c}_{32} & \tilde{b}_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where

$$\tilde{b}_n = \lambda_n b_n \rho_n^{-1},\ \tilde{a}_{nm} = \lambda_n a_{nm} \rho_m^{-1},\ \tilde{c}_{mn} = \lambda_m c_{mn} \rho_n^{-1}$$

with $\mathcal{R}^*$-valued sequences $(\lambda_n)_{n\in\mathbb{N}_0}$ and $(\rho_n)_{n\in\mathbb{N}_0}$. Then $w^{(N)}(w_0, \mathbf{H})$ is well-defined if and only if $w^{(N)}\left(\rho_0 w_0, \tilde{\mathbf{H}}\right)$ is well-defined. In this case,

$$w_n^{(N)}\left(\rho_0 w_0, \tilde{\mathbf{H}}\right) = \rho_n w_n^{(N)}(w_0, \mathbf{H})$$

for all $n \in \mathbb{N}_0$. Furthermore, $w(w_0, \mathbf{H})$ converges if and only if $w\left(\rho_0 w_0, \tilde{\mathbf{H}}\right)$ converges. In this case, for all $n \in \mathbb{N}_0$

$$w_n\left(\rho_0 w_0, \tilde{\mathbf{H}}\right) = \rho_n w_n(w_0, \mathbf{H}).$$

Note that the transformations we consider here can be written as

$$\tilde{\mathbf{H}} = T_\lambda \cdot \mathbf{H} \cdot T_\rho^{-1},$$

where T_λ and T_ρ are infinite-dimensional diagonal matrices (over $\mathcal{R}$) with entries $\lambda_0, \lambda_1, \ldots$ and $\rho_0, \rho_1, \ldots$ respectively. Solutions x of $\mathbf{H}x = 0$ then correspond to solutions $\tilde{x}$ of $\tilde{\mathbf{H}}\tilde{x} = 0$ by the simple transformation $\tilde{x} = T_\rho \cdot x$.

Intuitively, a generalization of equivalence transformations consists in replacing T_λ and T_ρ by invertible matrices. Invertibility of T_ρ is necessary due to the need of transforming both the matrix $\mathbf{H}$ and the solution x of $\mathbf{H}x = 0$. Invertibility of T_λ is reasonable since otherwise, equations of $\mathbf{H}x = 0$ can be eliminated, for example by allowing $\lambda_n = 0$ for some n in the case that T_λ is the diagonal matrix with entries $\lambda_0, \lambda_1, \ldots$.

In Theorem 4.1.1, we proved that $w^{(N)}$ solves the truncation of $\mathbf{H}x = 0$ (up to the boundary condition). More or less, this method corresponds to Gaussian elimination without swapping of positions of rows or columns. Replacing T_λ and T_ρ by arbitrary invertible matrices would allow this swapping. We will not focus on this topic, but we want to remark that allowing more general matrices T_λ and T_ρ makes things more complicated: The set of solutions of (4.1.2) may change, and even the values of $\operatorname{gcf}(\mathbf{H})$ or $w(w_0, \mathbf{H})$ may change essentially.

Example 4.3.1. Consider the situation of Example 4.2.1 again. We have seen that for any $w_0 \in \mathbb{C}$, the truncated system has solutions $\left(w_n^{(N)}\right)$ with $w_n^{(N)} = 2^n w_0$ for $n \leq N-1$ and $w_N^{(N)} = -2^N w_0$, converging to $(w_n)_{n\in\mathbb{N}_0}$ where $w_n = 2^n w_0$, but (unless $w_0 = 0$) this is no solution of the non-truncated system.

First, we want to emphasize on the fact that this behaviour is not due to the divergence of -2^N for $N \to \infty$. If applying the right-hand side transformation T_ρ^{-1} with $\rho_n = \frac{1}{4^n}$, the solution is rescaled by T_ρ, yielding solutions in which the factor 2 is replaced by $\frac{1}{2}$.

Now, we want to apply a left-side transformation to

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 1 & \cdots \\ & 2 & 1 & 1 & \cdots \\ & & 2 & 1 & \cdots \\ & & & \ddots & \ddots \end{pmatrix}.$$

The 0-row represents that we have no boundary condition, and therefore any solution of the sum equation

$$-2x_{n-1} = \sum_{m=n}^{\infty} x_m, \quad n \in \mathbb{N}$$

is a solution of the system $\mathbf{H}x = 0$.

In Example 4.2.1, we have already derived that there is no solution but the trivial one, $x = (0)$. Now, we consider $\tilde{\mathbf{H}} = T_\lambda \cdot \mathbf{H}$, where for T_λ we choose the non-diagonal matrix

$$T_\lambda = \begin{pmatrix} 1 & & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & \ddots \\ & & & & \ddots \end{pmatrix} \quad \text{with} \quad T_\lambda^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & 1 & 1 & \cdots \\ & & 1 & 1 & \cdots \\ & & & 1 & \cdots \\ & & & & \ddots \end{pmatrix},$$

and obtain

$$\tilde{\mathbf{H}} = T_\lambda \cdot \mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 2 & -1 & & & \\ & 2 & -1 & & \\ & & 2 & -1 & \\ & & & \ddots & \ddots \end{pmatrix},$$

and obviously, the system $\tilde{\mathbf{H}}\tilde{x} = 0$ is solved by $(\tilde{x}_n)_{n\in\mathbb{N}_0} = (2^n \tilde{x}_0)_{n\in\mathbb{N}_0}$ for any choice of $\tilde{x}_0 \in \mathbb{C}$. Still, this solution can be obtained by backward computation, and hence, it coincides with the gcf-based solution $w\left(w_0, \tilde{\mathbf{H}}\right)$.

Thus, here, the transformation described by T_λ essentially changes the set of the solutions, whereas convergence of the gcf-based solution and its values are not changed. This will not

be true in general: For example, we can apply T_λ again to $\tilde{\mathbf{H}}$, and obtain

$$\tilde{\tilde{\mathbf{H}}} := T_\lambda \tilde{\mathbf{H}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & -3 & 1 & & & \\ & 2 & -3 & 1 & & \\ & & 2 & -3 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Hence, $\tilde{\tilde{\mathbf{H}}}x = 0$ yields a second-order difference equation, any solution $x = (x_n)$ has the form $x_n = 2^n x_0 + x_0'$ for some $x_0, x_0' \in \mathbb{C}$. As pointed out above, it is well-known that backward computation yields a subdominant solution (see section 8.1 and references therein), that is $x_n = x_0'$ for some $x_0' \in \mathbb{C}$.

We have to admit that T_λ^{-1} is an informal inverse, that is, we did not specify an algebra for infinitely large matrices. In particular, in the space of all infinitely large matrices with finite row sum norm (or finite column sum norm), the matrix

$$T_\lambda = \begin{pmatrix} 1 & & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & \ddots \\ & & & & \ddots \end{pmatrix}$$

is not invertible. However, the same is true for diagonal matrices T_λ with $\lambda_n \to 0$ for $n \to \infty$, and such equivalence transformations work perfectly. In order to avoid such discussions in what follows, we will only use diagonal transformation matrices, that is, equivalence transformations in the sense of Theorem 4.3.1.

4.4 Convergence criteria involving S-series

When defining gcfs, we already mentioned that $L_{m,n,k}^{(N)}$ might have the interpretation $L_{m,n,k}^{(N)} = S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$. Due to $w_n = L_{n,1,0} w_0$, we can extend Theorem 2.3.1 to

Theorem 4.4.1. *Let* $\mathbf{H}$ *as in Definition* 2.3.1*, and define* $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$ *by* $p_{nn} = I - b_n$*,* $p_{mn} = -a_{mn}$*,* $p_{nm} = c_{nm}$ *for* $m < n$*.*

- *Let* $S(\mathbf{P}, n, 0, \mathbb{N})$ *converge unconditionally for all* $n \in \mathbb{N}_0$*,*
- *let* $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ *converge unconditionally for all* $n, N \in \mathbb{N}$*,* $n \le N$*, and*
- *let* $S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ *converge unconditionally for all* $k, n, m, N \in \mathbb{N}_0$ *with* $k < n \le m \le N$*.*

Then $\operatorname{gcf}(\mathbf{H})$ *converges with* $\operatorname{gcf}(\mathbf{H}) = I - S(\mathbf{P}, 0, 0, \mathbb{N})$*,* $w(w_0, \mathbf{H})$ *converges with* $w_n(w_0, \mathbf{H}) = S(\mathbf{P}, n, 0, \mathbb{N}) w_0$ *for* $n \ge 1$*, and* $w(w_0, \mathbf{H})$ *solves* (4.1.2)*. In case that* $\operatorname{gcf}(\mathbf{H})\, w_0 = 0$*,* $w(w_0, \mathbf{H})$ *solves* $\mathbf{H}x = 0$*.*

Proof. All conditions of Theorem 2.3.1 are fulfilled, hence $\operatorname{gcf}(\mathbf{H}) = I - S(\mathbf{P}, 0, 0, \mathbb{N})$ converges. Furthermore, $L^{(N)}_{m,n,k} = S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ is clear, yielding $L^{(N)}_{n,1,0} = S(\mathbf{P}, n, 0, \mathbb{N}_{1,N})$, and hence, $L_{n,1,0} = S(\mathbf{P}, n, 0, \mathbb{N})$ converges. For any $w_0 \in \mathcal{R}$, we obtain

$$\begin{aligned}
& -\sum_{m=0}^{n-1} c_{nm} w_m(w_0, \mathbf{H}) + b_n w_n(w_0, \mathbf{H}) + \sum_{m=n+1}^{\infty} a_{nm} w_m(w_0, \mathbf{H}) \\
= & \left(-p_{n0} - \sum_{m=1}^{n-1} p_{nm} S(\mathbf{P}, m, 0, \mathbb{N}) + (I - p_{nn}) S(\mathbf{P}, n, 0, \mathbb{N}) - \sum_{m=n+1}^{\infty} p_{nm} S(\mathbf{P}, m, 0, \mathbb{N}) \right) w_0 \\
= & \left(S(\mathbf{P}, n, 0, \mathbb{N}) - p_{n0} - \sum_{m=1}^{\infty} p_{nm} S(\mathbf{P}, m, 0, \mathbb{N}) \right) w_0 \\
= & \left(S(\mathbf{P}, n, 0, \mathbb{N}) - S(\mathbf{P}, n, 0, \mathbb{N}) \right) w_0 = 0
\end{aligned}$$

for $n \geq 1$, that is, $w(w_0, \mathbf{H})$ solves (4.1.2). For solving $\mathbf{H}x = 0$, the initial condition (4.1.1) has to be satisfied, that is

$$\begin{aligned}
0 &= b_0 w_0 + \sum_{m=1}^{\infty} a_{0m} w_m(w_0, \mathbf{H}) \\
&= \left(I - p_{00} - \sum_{m=1}^{\infty} p_{0m} S(\mathbf{P}, m, 0, \mathbb{N}) \right) w_0 \\
&= \left(I - S(\mathbf{P}, 0, 0, \mathbb{N}) \right) w_0 \\
&= \operatorname{gcf}(\mathbf{H})\, w_0.
\end{aligned}$$

□

4.5 Pringsheim-type convergence criteria for w

Now, we want to state Pringsheim-type criteria for convergence of w to a solution of $\mathbf{H}x = 0$.

In Theorem 3.2.3, we stated a Pringsheim-type criterion for convergence of $\operatorname{gcf}(\mathbf{H})$ which involved Pringsheim-Lyapunov functions. The statement was based on a real-valued matrix $\mathbf{U}$ obtained by performing equivalence transformations, taking norms and transposing the real-valued matrix $\hat{\mathbf{P}}$. Note that in the proof of Theorem 3.2.3, we showed (unconditional) convergence of $S(\mathbf{U}, n, 0, \mathbb{N})$. If $\mathbf{U}$ can be obtained from $\mathbf{P}$ without transposition, this yields unconditional convergence of $S(\mathbf{P}, n, 0, \mathbb{N})$, that is, the conditions of Theorem 4.4.1 are fulfilled. Thus, we obtain

Theorem 4.5.1. *Let $\hat{\mathbf{P}}$ be defined as in section 3.2, and let there be a Pringsheim-Lyapunov function for $\hat{\mathbf{P}}$. Then $\operatorname{gcf}(\mathbf{H})$ and $w(w_0, \mathbf{H})$ converge for any choice of w_0. $w(w_0, \mathbf{H})$ solves (4.1.2), and in case of $\operatorname{gcf}(\mathbf{H})\, w_0 = 0$, $w(w_0, \mathbf{H})$ solves $\mathbf{H}x = 0$.*

Again, condition (PL3) for Pringsheim-Lyapunov functions can be replaced by simpler ones according to Theorem 3.2.4. However, using the transposed matrix $\hat{\mathbf{P}}^T$ will yield more elegant results. When replacing $\hat{\mathbf{P}}$ by its transpose, we obtain convergence of $S(\mathbf{P}, 0, n, \mathbb{N})$ instead of

$S(\mathbf{P}, n, 0, \mathbb{N})$. Indeed, if $\hat{\mathbf{P}}^T$ admits a Pringsheim-Lyapunov function, w might still diverge. Hence, we define

Definition 4.5.1. Let $\mathbf{U} = (u_{ij})_{i,j\in\mathbb{N}_0}$ be a $\mathbb{R}_{\geq 0}$-valued matrix. A function $g : \mathbb{N}_0 \to \mathbb{R}$, $j \mapsto g_j$ is said to be a *strong Pringsheim-Lyapunov function* for $\mathbf{U}$ if

(PL0) $g_j > 0$ for all $j \in \mathbb{N}_0$ and $g_0 = 1$.

(PL1) $\sum_{k=0}^{\infty} u_{0k} g_k < \infty$.

(PL2) $\sum_{k=0}^{\infty} u_{mk} g_k \leq g_m$ for all $m \in \mathbb{N}$.

(PL4) for all $i \in \mathbb{N}$ there is some $j \in \mathbb{N}$ with

- $i \overset{\mathbf{U}_{1,\infty}}{\to} j$ and
- $\sum_{k=1}^{\infty} u_{jk} g_k < g_j$.

Before we use strong Pringsheim-Lyapunov functions for proving a Pringsheim-type convergence criterion for w, we give a reformulation of (PL4) which is quite similar to (PL3), and we give a sufficient condition for (PL4) which is very similar to the criterion for (PL3) in Theorem 3.2.4.

Theorem 4.5.2. *Let $\mathbf{U} = (u_{ij})_{i,j\in\mathbb{N}_0}$ be a $\mathbb{R}_{\geq 0}$-valued matrix, and let g meet (PL0), (PL1) and (PL2).*

- *g meets (PL4) if and only if for all $n \in \mathbb{N}$, $i \in \mathbb{N}_{n,\infty}$, there is some $j \in \mathbb{N}_{n,\infty}$ such that*
  - *$i \overset{\mathbf{U}_{n,\infty}}{\to} j$ and*
  - *$\sum_{k=n}^{\infty} u_{jk} g_k < g_j$.*
- *In case that*

(PL4a) for all $n \in \mathbb{N}$ there is $m < n$ with $u_{nm} > 0$

holds, g is a strong Pringsheim-Lyapunov function.

Proof. • The conditions of the statement clearly imply (PL4) (put $n = 1$). On the other hand, let (PL4) hold, and let $i \geq n$. Due to (PL4), there is some $j_1 \in \mathbb{N}$ such that $\sum_{k=1}^{\infty} u_{j_1 k} g_k < g_{j_1}$. If $j_1 \geq n$ and $i \overset{\mathbf{U}_{n,\infty}}{\to} j$, the statement is true with $j = j_1$. Otherwise, there is some $j \geq n$ with

- $i \overset{\mathbf{U}_{n,\infty}}{\to} j$ and

– $u_{j\ell} > 0$ for some $\ell < n$, and hence $\sum_{k=n}^{\infty} u_{jk} g_k \leq g_j - u_{j\ell} g_\ell < g_j$.

- Analogously to Theorem 3.2.4.

□

With this new term, we prove

Theorem 4.5.3. *Let $\hat{\mathbf{P}}$ be defined as in section 3.2, nd let there be a strong Pringsheim-Lyapunov function for $\mathbf{U} = \hat{\mathbf{P}}^T$. Then* gcf $(\mathbf{H})$ *and $w(w_0, \mathbf{H})$ converge for any choice of w_0. $w(w_0, \mathbf{H})$ solves* (4.1.2)*, and in case of* gcf $(\mathbf{H})\, w_0 = 0$*, $w(w_0, \mathbf{H})$ solves $\mathbf{H}x = 0$.*

Proof. Additionally to the proof of Theorem 3.2.3, we have to prove that $S(\mathbf{P}, n, 0, \mathbb{N})$ converges absolutely (and hence, unconditionally). By submultiplicativity of the norm, convergence of $S(\mathbf{U}, 0, n, \mathbb{N})$ with $\mathbf{U} = \hat{\mathbf{P}}^T$ suffices. The convergence of $S(\mathbf{U}, 0, n, \mathbb{N})$ is proved by considerations quite similar to those in the proof of Theorem 3.2.3.

- We prove $S_L(\mathbf{U}, m, n, \mathbb{N} \setminus \{n\}) g_n \leq g_m$ for all $m \in \mathbb{N}_0$ by induction with respect to L. For $L = 1$, the statement is trivial due to $u_{mn} g_n \leq g_m$, and for $L \geq 2$, we have

$$\begin{aligned} S_L(\mathbf{U}, m, n, \mathbb{N} \setminus \{n\}) g_n &= u_{mn} g_n + \sum_{\substack{j=1 \\ j \neq n}}^{\infty} u_{mj} S_{L-1}(\mathbf{U}, j, n, \mathbb{N} \setminus \{n\}) \\ &\leq u_{mn} g_n + \sum_{\substack{j=1 \\ j \neq n}}^{n} u_{mj} g_j = \sum_{j=1}^{\infty} u_{mj} g_j \leq g_m. \end{aligned}$$

  In particular, for $N \to \infty$, we obtain $S(\mathbf{U}, 0, n, \mathbb{N} \setminus \{n\}) g_n \leq g_0$, and hence,

$$S(\mathbf{U}, 0, n, \mathbb{N} \setminus \{n\}) \leq \frac{1}{g_n} < \infty.$$

- More precisely, we obtain

$$S(\mathbf{U}, m, n, \mathbb{N} \setminus \{n\}) g_n \leq \sum_{k=1}^{\infty} u_{mk} g_k.$$

  Now choose $j \in \mathbb{N}$ according to (PL4). Then $S(\mathbf{U}, j, n, \mathbb{N} \setminus \{n\}) < g_j$. If $u_{ij} > 0$ for some $i \in \mathbb{N}$, we obtain

$$S(\mathbf{U}, i, n, \mathbb{N} \setminus \{n\}) g_n = u_{in} g_n + \sum_{\substack{k=1 \\ k \neq n}}^{\infty} u_{ik} S(\mathbf{U}, k, n, \mathbb{N} \setminus \{n\}) g_n \leq \sum_{k=1}^{\infty} u_{ik} g_k < g_i.$$

  After finitely many iterations (we have $n \to j$), we derive $S(\mathbf{U}, n, n, \mathbb{N} \setminus \{n\}) g_n < g_n$, and thus, $S(\mathbf{U}, n, n, \mathbb{N}) = \frac{1}{1 - S(\mathbf{U}, n, n, \mathbb{N} \setminus \{n\})} - 1 < \infty$.

- Summarizing, we obtain

$$S(\mathbf{U}, 0, n, \mathbb{N}) = S(\mathbf{U}, 0, n, \mathbb{N} \setminus \{n\})(1 + S(\mathbf{U}, n, n, \mathbb{N})) < \infty,$$

  and from the second consideration, we directly obtain

$$S(\mathbf{U}, n, n, \mathbb{N}_{n,N}) \leq S(\mathbf{U}, n, n, \mathbb{N}) < \infty.$$

- Finally, for $k < n \leq m \leq N$, we have

$$S(\mathbf{U}, k, m, \mathbb{N}_{n,N}) \leq S(\mathbf{U}, k, m, \mathbb{N} \setminus \{m\}) \cdot (1 + S(\mathbf{U}, m, m, \mathbb{N})) < \infty,$$

  yielding unconditional convergence of $S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$.

□

4.6 Inhomogeneous systems

Up to now, we studied the homogeneous system $\mathbf{H}x = 0$. In this section, we briefly demonstrate how to extend our results to inhomogeneous systems. These considerations are based on two observations:

- When putting $x_0 = I$, $(x_1, x_2, \ldots)^T$ automatically solves an inhomogeneous system.
- The sequence w is a candidate for solving $\mathbf{H}x = 0$ if w_0 is chosen as a solution of $\operatorname{gcf}(\mathbf{H})\, w_0 = 0$. If $\operatorname{gcf}(\mathbf{H}) = 0$, we may choose $w_0 = I$.

Hence, we put $b_0 = 0$ and $a_{0n} = 0$ for all $n \in \mathbb{N}$. Then, the initial condition (4.1.1) of the system $\mathbf{H}x = 0$ becomes trivial, and with $x_0 = I$, we may rewrite $\mathbf{H}x = 0$ as

$$c_{n0} + \sum_{m=1}^{n-1} c_{nm} x_m = b_n x_n + \sum_{m=n+1}^{\infty} a_{nm} x_m, \qquad n \in \mathbb{N}, \tag{4.6.1}$$

and this is the general form of an inhomogeneous system of linear equations in $\mathcal{R}$. A gcf-based candidate for a solution is given by $w^{\mathrm{inh}}(\mathbf{H}) := (L_{n,1,0})_{n \in \mathbb{N}}$. Note that the definition of $L_{n,1,0}$ is not influenced by b_0 or a_{0n}.

In Theorem 4.4.1, we gave a S-series criterion for w solving $\mathbf{H}x = 0$, and we can directly state the same result here, where we just ignore b_0, a_{0n} and the condition $\operatorname{gcf}(\mathbf{H})\, x_0 = 0$.

Theorem 4.6.1. *Define $\mathbf{P} = (p_{ij})_{i,j \in \mathbb{N}_0}$ by $p_{nn} = I - b_n$ for $n > 0$, $p_{mn} = -a_{mn}$ for $0 < m < n$, and $p_{nm} = c_{nm}$ for $m < n$.*

- *Let $S(\mathbf{P}, n, 0, \mathbb{N})$ converge unconditionally for all $n \in \mathbb{N}$,*
- *let $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ converge unconditionally for all $n, N \in \mathbb{N}$, $n \leq N$, and*
- *let $S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ converge unconditionally for all $k, n, m, N \in \mathbb{N}_0$ with $k < n \leq m \leq N$.*

Then $w^{inh}(\mathbf{H})$ *converges and solves* (4.6.1).

Naturally, Pringsheim-type criteria can be stated for inhomogeneous systems, too. Since we ignore the initial condition, Theorem 4.5.3 directly yields

Theorem 4.6.2. *With* $a_{0n} = 0$*, let* $\hat{\mathbf{P}}$ *be defined as in section* 3.2*, and let there be a strong Pringsheim-Lyapunov function for* $\hat{\mathbf{P}}^T$*. Then* w^{inh} *converges and solves* (4.6.1).

4.7 Transposed systems

Instead of considering the system $\mathbf{H}x = 0$, we might be interested in solving $\psi\mathbf{H} = 0$, that is

$$\psi_0 b_0 = \sum_{m=1}^{\infty} \psi_m c_{m0} \tag{4.7.1}$$

$$\sum_{m=0}^{n-1} \psi_m a_{mn} + \psi_n b_n = \sum_{m=n+1}^{\infty} \psi_m c_{mn}, \qquad n \in \mathbb{N}. \tag{4.7.2}$$

The inhomogeneous variant is given by

$$a_{0n} + \sum_{m=1}^{n-1} \psi_m a_{mn} + \psi_n b_n = \sum_{m=n+1}^{\infty} \psi_m c_{mn}, \qquad n \in \mathbb{N}. \tag{4.7.3}$$

Unsurprisingly, we define $v = (v_n)_{n\in\mathbb{N}_0}$ where $v_n = v_0 R_{0,1,n}$ for $n \geq 1$ and $v_n^{\text{inh}} = R_{0,1,n}$. Without restating the proofs, we give the most important results.

Theorem 4.7.1 (Compare with Theorem 4.4.1)**.** *Let* $\mathbf{H}$ *as in Definition* 2.3.1*, and define* $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$ *by* $p_{nn} = I - b_n$*,* $p_{mn} = -a_{mn}$*,* $p_{nm} = c_{nm}$ *for* $m < n$*.*

- *Let* $S(\mathbf{P}, 0, n, \mathbb{N})$ *converge unconditionally for all* $n \in \mathbb{N}_0$*,*
- *let* $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ *converge unconditionally for all* $n, N \in \mathbb{N}$*,* $n \leq N$*, and*
- *let* $S(\mathbf{P}, k, m, \mathbb{N}_{n,N})$ *converge unconditionally for all* $k, n, m, N \in \mathbb{N}_0$ *with* $k < n \leq m \leq N$*.*

Then $\operatorname{gcf}(\mathbf{H})$ *converges with* $\operatorname{gcf}(\mathbf{H}) = I - S(\mathbf{P}, 0, 0, \mathbb{N})$*,* $v(v_0, \mathbf{H})$ *converges with* $v_n(v_0, \mathbf{H}) = v_0 S(\mathbf{P}, 0, n, \mathbb{N})$ *for* $n \geq 1$*, and* v *solves* (4.7.2)*. If additionally* $v_0 \operatorname{gcf}(\mathbf{H}) = 0$*,* v *solves* $\psi\mathbf{H} = 0$*.*

Theorem 4.7.2 (Compare with Theorem 4.5.3)**.** *Let* $\hat{\mathbf{P}}$ *be defined as in section* 3.2*, and let there be a strong Pringsheim-Lyapunov function for* $\hat{\mathbf{P}}$*. Then* $\operatorname{gcf}(\mathbf{H})$ *and* $v(v_0, \mathbf{H})$ *converge for any choice of* v_0*, and* $v(v_0, \mathbf{H})$ *solves* (4.7.2)*. In case of* $v_0 \operatorname{gcf}(\mathbf{H}) = 0$*,* $v(v_0, \mathbf{H})$ *solves* $\psi\mathbf{H} = 0$*.*

Theorem 4.7.3 (Compare with Theorem 4.6.1)**.** *Define* $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$ *by* $p_{nn} = I - b_n$ *(*$n > 0$*),* $p_{mn} = -a_{mn}$*,* $p_{nm} = c_{nm}$ *(*$m > 0$*) for* $m < n$*.*

- *Let* $S(\mathbf{P}, 0, n, \mathbb{N})$ *converge unconditionally for all* $n \in \mathbb{N}$*,*
- *let* $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ *converge unconditionally for all* $n, N \in \mathbb{N}$*,* $n \leq N$*, and*
- *let* $S(\mathbf{P}, k, m, \mathbb{N}_{n,N})$ *converge unconditionally for all* $k, n, m, N \in \mathbb{N}_0$ *with* $k < n \leq m \leq N$*.*

Then $v^{inh}(\mathbf{H})$ *converges and solves* (4.7.3)*.*

Theorem 4.7.4 (Compare with Theorem 4.6.2)**.** *With* $c_{n0} = 0$*, let* $\hat{\mathbf{P}}$ *be defined as in section* 3.2*, and let there be a strong Pringsheim-Lyapunov function for* $\hat{\mathbf{P}}$*. Then* v^{inh} *converges and solves* (4.7.3)*.*

4.8 Criteria using positivity and minimality

Up to now we only gave sufficient criteria for the gcf-based solutions $w(w_0, \mathbf{H})$ or $v(v_0, \mathbf{H})$ to converge and to meet $\mathbf{H}x = 0$ or $\psi\mathbf{H} = 0$ respectively. Naturally, infinite systems of equations may have more than one solution, and hence, a simple question arises: Which of these solutions is characterized by gcfs? In chapter 8, we will discuss whether or not it is possible to find an answer in terms of dominant and subdominant solutions. Here, we focus on minimality in the context of operator algebras with positive elements.

For a brief introduction to positivity on operator algebras, we refer to section B.3 in the appendix. If $p_{ij} \geq 0$ for all i, j, convergence of $S(\mathbf{P}, i, j, A)$ implies unconditional convergence (see Theorem B.3.1). Hence, if $p_{nn} = I - \lambda_n b_n \rho_n^{-1}$, $p_{mn} = -\lambda_m a_{mn} \rho_n^{-1}$, $p_{nm} = \lambda_n c_{nm} \rho_m^{-1}$ (for $m < n$) yield $p_{mn} \geq 0$ for all $m, n \in \mathbb{N}_0$, we are allowed to replace unconditional convergence of the S-series by mere convergence in Theorems 2.3.1, 4.4.1 and 4.7.1.

Now we prove the property of minimality of $w(w_0, \mathbf{H})$. As in section B.3, we let $\mathcal{R}$ be an operator algebra with positive cone $\mathcal{R}_+$.

Theorem 4.8.1. *Let* $\mathbf{H}$ *as in Definition* 2.3.1*, let* (λ_n) *and* (ρ_n) *be* $\mathcal{R}^* \cap \mathcal{R}_+$*-valued sequences. Define* $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$ *by* $p_{nn} = I - \lambda_n b_n \rho_n^{-1}$*,* $p_{mn} = -\lambda_m a_{mn} \rho_n^{-1}$*,* $p_{nm} = \lambda_n c_{nm} \rho_m^{-1}$ *for* $m < n$*, and let* $p_{mn} \in \mathcal{R}_+$ *for all* $m, n \in \mathbb{N}_0$*.*

- *Let* $w_0 \in \mathcal{R}_+$ *and*
  - *let* $S(\mathbf{P}, n, 0, \mathbb{N})$ *converge for all* $n \in \mathbb{N}_0$*,*
  - *let* $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ *converge for all* $n, N \in \mathbb{N}$*,* $n \leq N$*, and*
  - *let* $S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ *converge for all* $k, n, m, N \in \mathbb{N}_0$ *with* $k < n \leq m \leq N$*.*

  Then $\operatorname{gcf}(\mathbf{H})$ *and* $w(w_0, \mathbf{H})$ *converge and* $w = w(w_0, \mathbf{H})$ *is the minimal solution of* $\mathbf{H}x \geq 0$ *(componentwise),* $x \geq 0$ *(componentwise) subject to* $x_0 = w_0$*.*
- *Let* $v_0 \in \mathcal{R}_+$ *and*

– *let* $S(\mathbf{P}, 0, n, \mathbb{N})$ *converge for all* $n \in \mathbb{N}_0$,

– *let* $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ *converge for all* $n, N \in \mathbb{N}$, $n \leq N$, *and*

– *let* $S(\mathbf{P}, k, m, \mathbb{N}_{n,N})$ *converge for all* $k, n, m, N \in \mathbb{N}_0$ *with* $k < n \leq m \leq N$.

Then $\operatorname{gcf}(\mathbf{H})$ *and* $v(v_0, \mathbf{H})$ *converge and* $v = v(v_0, \mathbf{H})$ *is the minimal solution of* $\psi\mathbf{H} \geq 0$, $\psi \geq 0$ *subject to* $\psi_0 = v_0$.

In both cases, strict inequality can only hold in the initial condition of $\mathbf{H}x \geq 0$ *or* $\psi\mathbf{H} \geq 0$ *respectively.*

Proof. We consider the first point here, the proof for the second one runs as well.

Convergence of w is implied by Theorem 4.4.1 and positivity of p_{ij} (for all i, j which are used in the S-series). Due to the definition of $\mathbf{P}$ and $\lambda_n, \rho_n \in \mathcal{R}_+$, $\mathbf{H}x \geq 0$ is equivalent to $\mathbf{P}x \leq x$.

According to the definition of S-series, we have

$$w_n = L_{n,1,0}w_0 = S(\mathbf{P}, n, 0, \mathbb{N})w_0 = \sum_{\substack{m=1 \\ i_0=n, i_1, \ldots, i_{m-1} \geq 1, i_m = 0}}^{\infty} \prod_{r=1}^{m} p_{i_{r-1}, i_r} w_0 \geq 0$$

for $n \geq 1$. Now we define

$$w_{n,M} := \sum_{\substack{m=1 \\ i_0=n, i_1, \ldots, i_{m-1} \geq 1, i_m = 0}}^{M} \prod_{r=1}^{m} p_{i_{r-1}, i_r} w_0 \geq 0$$

for $n, M \geq 1$ and $w_{0,M} = w_0$ for all $M \geq 1$, we assume that $x\mathbf{P} \leq x$, $x \geq 0$ and $x_0 = w_0$. Then $w_{0,M} = x_0$, and we prove $w_{n,M} \leq x_n$ for $n \geq 1$ by induction with respect to M. For $M = 1$, we have

$$w_{n,1} = p_{n0}w_0 = p_{n0}x_0 \leq \sum_{k=0}^{\infty} p_{nk}x_k \leq x_n.$$

Assuming $w_{k,M} \leq x_k$ for all $k \in \mathbb{N}$ yields

$$\begin{aligned} w_{n,M+1} &= p_{n0}w_0 + \sum_{k=1}^{\infty} \sum_{\substack{m=2 \\ i_0=n, i_1=k, i_2, \ldots, i_{m-1} \geq 1, i_m = 0}}^{M+1} \prod_{r=1}^{m} p_{i_{r-1}, i_r} w_0 \\ &= p_{n0}w_0 + \sum_{k=1}^{\infty} p_{nk} \sum_{\substack{m=1 \\ i_0=k, i_1, \ldots, i_{m-1} \geq 1, i_m = 0}}^{M} \prod_{r=1}^{m} p_{i_{r-1}, i_r} w_0 \\ &= p_{n0}w_0 + \sum_{k=1}^{\infty} p_{nk}w_{k,M} \leq \sum_{k=0}^{\infty} p_{nk}x_k \leq x_n. \end{aligned}$$

Finally, $w_n = \lim_{N \to \infty} w_{n,M}$ yields $w_n \leq x_n$ for all $n \in \mathbb{N}_0$. □

We have discussed inhomogeneous systems by ignoring the initial condition (that is, by replacing the appropriate coefficients by 0), and setting $x_0 = I$. Hence, the initial condition is always met, and in case of convergence, we obtain a solution of the inhomogeneous system. Under the conditions of Theorem 4.8.1, the gcf-based solution is minimal again.

Using Lyapunov functions, it seems quite natural to develop Pringsheim-type criteria again. In fact, the proof of Theorem 3.2.3 can be applied directly to $\mathbf{P}$ instead of $\mathbf{U}$ if replacing $g_n \in (0,\infty)$ by $g_n \in \mathcal{R}^* \cap \mathcal{R}_+$. However, in the situation of operator algebras, it is more natural to use Lyapunov function $n \mapsto g_n \in F$ instead of $g_n \in \mathcal{R} = L(F)$. Then things get more complicated. Thus, we omit Pringsheim-Lyapunov type criteria, but we state a speed-of-convergence estimate which is similar to that in Theorem 3.2.3.

Theorem 4.8.2. *Define p_{mn} as in Theorem* 4.8.1*, let the conditions of Theorem* 2.3.1 *be met, let $p_{mn} \geq 0$ for all $m, n \in \mathbb{N}_0$, and let there be a function $n \mapsto g_n \in F_+$ with*

$$\sum_{k=0}^{\infty} p_{mk} g_k \leq g_m, \qquad m \in \mathbb{N}$$

(that is, (PL2)). Then for the speed of convergence, we have

$$0 \leq \lambda_0 \left(K_0^{(N)} - \operatorname{gcf}(\mathbf{H}) \right) \rho_0^{-1} g_0 \leq \sum_{m=N+1}^{\infty} S(\mathbf{P}, 0, m, \mathbb{N}_{1,N}) g_m.$$

Proof. We know that

$$\lambda_0 \left(K_0^{(N)} - \operatorname{gcf}(\mathbf{H}) \right) \rho_0^{-1} = S(\mathbf{P}, 0, 0, \mathbb{N}) - S(\mathbf{P}, 0, 0, \mathbb{N}_{1,N}) \geq 0,$$

and due to $g_0 \geq 0$, the first inequality is clear. For the second one, we find

$$S_L(\mathbf{P}, m, 0, \mathbb{N}) g_0 \leq g_m \qquad \Rightarrow \qquad S(\mathbf{P}, m, 0, \mathbb{N}) g_0 \leq g_m$$

with the same induction as in the proof of Theorem 3.2.3. With the decomposition

$$S(\mathbf{P}, 0, 0, \mathbb{N}) - S(\mathbf{P}, 0, 0, \mathbb{N}_{1,N}) = \sum_{m=N+1}^{\infty} S(\mathbf{P}, 0, m, \mathbb{N}_{1,N}) S(\mathbf{P}, m, 0, \mathbb{N}),$$

the statement follows. □

4.9 Systems with block matrices

A general assumption for our results is that a_{mn}, b_n, c_{nm} are all elements of the same Banach algebra $\mathcal{R}$. A quite typical situation is $\mathcal{R} = \mathbb{C}^{d \times d}$, that is, we want to solve an (infinite) system of equations in $\mathbb{C}$, and we have a block-matrix representation. Often, systems are written in this manner in order to obtain a certain structure (block-tridiagonal matrices, block-Hessenberg matrices, ...).

The assumption that all blocks have the same dimension is not natural: In principle, we could choose an arbitrary partition of the set of equations. Here, we will only consider the case where each partition consists of finitely many equations, that is, we want to solve the equations (4.1.2) and (4.1.1) where $a_{mn} \in \mathbb{C}^{d_m \times d_n}, b_n \in \mathbb{C}^{d_n \times d_n}$, $c_{nm} \in \mathbb{C}^{d_n \times d_m}$ and $x_n \in \mathbb{C}^{d_n}$.

Formally, a_{mn}, b_n, c_{nm} are no elements of one Banach algebra. Nevertheless, it is natural to try to define $L_{n,1,0}$ and w_n using the coefficients a_{mn}, b_n, c_{nm} since 'all dimensions fit together'. In order to find a mathematical foundation for this approach, we consider the Banach algebra

$$\mathcal{R} = \left\{ Z = (z_{ij})_{i,j\in\mathbb{N}} : \ \ ||Z|| := \sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |z_{ij}| < \infty \right\},$$

where $I = (\delta_{ij})_{i,j\in\mathbb{N}}$. By putting

$$\tilde{b}_n = \begin{pmatrix} b_n & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}, \qquad \tilde{a}_{mn} = \begin{pmatrix} a_{mn} & \\ & 0 \end{pmatrix}, \qquad \tilde{c}_{nm} = \begin{pmatrix} c_{nm} & \\ & 0 \end{pmatrix},$$

we embed the coefficients in $\mathcal{R}$. Simple induction yields

$$\tilde{K}_n^{(N)} = \begin{pmatrix} K_n^{(N)} & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}, \quad \tilde{L}_{m,n,k}^{(N)} = \begin{pmatrix} L_{m,n,k}^{(N)} & \\ & 0 \end{pmatrix}, \quad \tilde{R}_{k,n,m}^{(N)} = \begin{pmatrix} R_{k,n,m}^{(N)} & \\ & 0 \end{pmatrix},$$

that is

- $\tilde{K}_n^{(N)} \in \mathcal{R}^*$ if and only if $K_n^{(N)}$ is an invertible matrix,
- $\operatorname{gcf}\left(\tilde{\mathbf{H}}\right)$ converges if and only if $\operatorname{gcf}(\mathbf{H})$ converges, and in case of convergence, we have

$$\operatorname{gcf}\left(\tilde{\mathbf{H}}\right) = \begin{pmatrix} \operatorname{gcf}(\mathbf{H}) & \\ & 0 \end{pmatrix}.$$

- $\operatorname{gcf}\left(\tilde{\mathbf{H}}\right) \tilde{w}_0 = 0$ if and only if $\tilde{w}_0 = \binom{w_0}{\dots}$ where $w_0 \in \mathbb{R}^{d_0}$ with $\operatorname{gcf}(\mathbf{H})\, w_0 = 0$.
- $\tilde{v}_0 \operatorname{gcf}\left(\tilde{\mathbf{H}}\right) = 0$ if and only if $\tilde{v}_0 = (v_0, \dots)$ where $v_0 \in \mathbb{R}^{d_0}$ with $v_0 \operatorname{gcf}(\mathbf{H}) = 0$.
- $w\left(\tilde{w}_0, \tilde{\mathbf{H}}\right)$ converges if and only if $w(w_0, \mathbf{H})$ converges, and in case of convergence, $w_n\left(\tilde{w}_0, \tilde{\mathbf{H}}\right) = \binom{w_n(w_0, \mathbf{H})}{0}$ for $n \geq 1$.
- $v\left(\tilde{v}_0, \tilde{\mathbf{H}}\right)$ converges if and only if $v(v_0, \mathbf{H})$ convergs, and in case of convergence, we have $v_n\left(\tilde{v}_0, \tilde{\mathbf{H}}\right) = (v_n(v_0, \mathbf{H}), 0)$ for $n \geq 1$.

Hence, we can still use our usual definitions when using the block matrices a_{mn}, b_n, c_{nm}. Furthermore, all S-series convergence criteria hold: Let I_{d_n} be the $d_n \times d_n$-unity matrix, let $p_{nn} = I_{d_n} - b_n$, $p_{mn} = -a_{mn}$ and $p_{nm} = c_{nm}$ (for $m < n$) and $\tilde{p}_{nn} = I - \tilde{b}_n$, $\tilde{p}_{mn} = -\tilde{a}_{mn}$ and $\tilde{p}_{nm} = \tilde{c}_{nm}$ (for $m < n$). Then we have

$$\tilde{p}_{mn} = \begin{pmatrix} p_{mn} & \\ & 0 \end{pmatrix}.$$

for all $m, n \in \mathbb{N}_0$, and hence, for all $i, j \in \mathbb{N}_0$ and $A \subset \mathbb{N}_0$, we obtain

$$S\left(\tilde{\mathbf{P}}, i, j, A\right) = \begin{pmatrix} S(\mathbf{P}, i, j, A) & \\ & 0 \end{pmatrix},$$

that is $S\left(\tilde{P}, i, j, A\right)$ converges unconditionally if and only if $S(\mathbf{P}, i, j, A)$ does.

In total, all results of chapter 3 and of chapter 4 still hold if the coefficients are $d_m \times d_n$-matrices. Similar considerations hold for more structures, see section 5.1 for a slight generalization.

4.10 GCFs and Gaussian elimination

For truncation index $N+1$, the truncated system (4.1.4) becomes

$$\sum_{m=0}^{n-1} c_{nm} x_m = b_n x_n + \sum_{m=n+1}^{N+1} a_{nm} x_m, \qquad n = 1, \ldots, N+1.$$

The basic idea of Gaussian elimination is to add multiples of one of these equations to the other equations in such a way that the coefficients of some x_k become 0. In particular, we can use the $(N+1)$st equation for replacing the coefficients $a_{n,N+1}$ by 0. The $(N+1)$st equation can be written as

$$x_{N+1} = \sum_{m=0}^{N} b_{N+1}^{-1} c_{N+1,m} x_m, \tag{4.10.1}$$

and by multiplying (4.10.1) with $-a_{n,N+1}$ and adding it to the Nth equation, we obtain

$$\begin{aligned} \sum_{m=0}^{n-1} \left(c_{nm} - a_{n,N+1} b_{N+1}^{-1} c_{N+1,m}\right) x_m &= \left(b_n + a_{n,N+1} b_{N+1}^{-1} c_{N+1,n}\right) x_n \\ &\quad + \sum_{m=n+1}^{N} \left(a_{nm} + a_{n,N+1} b_{N+1}^{-1} c_{N+1,m}\right) x_m. \end{aligned}$$

Using the coefficients $\tilde{c}_{nm}, \tilde{b}_n, \tilde{a}_{nm}$ as defined in the proof of Theorem 4.1.1, we can write

$$\sum_{m=0}^{n-1} \tilde{c}_{nm} x_m = \tilde{b}_n x_n + \sum_{m=n+1}^{N} \tilde{a}_{nm} x_m,$$

and this corresponds to the truncation of $\tilde{\mathbf{H}}x = 0$ at index N. Furthermore, (4.10.1) can be written as

$$x_{N+1} = \sum_{m=0}^{N} L^{(N+1)}_{N+1,N+1,m} x_m,$$

which is equivalent (simple induction) to $x_{N+1} = L^{(N+1)}_{N+1,1,0} x_0$. Hence, a step of Gaussian elimination exactly corresponds to the induction step in the proof of Theorem 4.1.1, and the definition of $w^{(N)}$ corresponds to forward substitution. In total, for finite matrices, the gcf–approach turns out to be the backward variant of LU decomposition without pivoting, which we will refer to as BackLU.

With our notation, it is obvious that BackLU works if and only if $K_N^{(N)}, \ldots, K_1^{(N)} \in \mathcal{R}^*$, and we can use our theory to state criteria for BackLU to work. Since the focus of our work is not on solving finite systems of equations, but infinite ones, we only give a general criterion in terms of S-series, and for tridiagonal matrices, we state a Pringsheim–type criterion which can be obtained from Theorem 3.3.5.

Theorem 4.10.1. *Define $\mathbf{P} = (p_{ij})_{i,j=0}^{N}$ by $p_{nn} = I - b_n$, $p_{mn} = -a_{mn}$, $p_{nm} = c_{nm}$ for $m < n$ and let*

- *$S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ converge unconditionally for all $n \in \{1, \ldots, N\}$,*
- *$S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ converge unconditionally for all $k, n, m \in \{1, \ldots, N\}$ with $k < n \leq m$.*

Then $\mathbf{H}_{0,N}x = 0$ and $\psi\mathbf{H}_{0,N} = 0$ can be solved by BackLU.

Theorem 4.10.2. *Consider the tridiagonal matrix*

$$\mathbf{H}_{0,N} = \begin{pmatrix} b_0 & a_{01} & & & & & \\ -c_{10} & b_1 & a_{12} & & & & \\ & -c_{21} & b_2 & a_{23} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -c_{N-1,N_2} & b_{N-1} & a_{N-1,N} \\ & & & & -c_{N,N-1} & b_n \end{pmatrix},$$

let $b_n^{-1} \in \mathcal{R}^*$ for all $n = 1, \ldots, N$ and let

- *$a_{n,n+1} \neq 0$ for all $n = 0, \ldots, N-1$ or*
- *$c_{n+1,n} \neq 0$ for all $n = 0, \ldots, N-1$.*

Furthermore, let $(\lambda_n)_{n=0}^{N}$ and $(\rho_n)_{n=0}^{N}$ be $\mathcal{R}^*$-valued sequences, let $(\alpha_n)_{n=0}^{N}$ be defined by $\alpha_1 = \left|\left|\lambda_0 a_{01} b_1^{-1} \lambda_1^{-1}\right|\right| \cdot \left|\left|\lambda_1 c_{10} \rho_0^{-1}\right|\right|$ and

$$\alpha_n = \left|\left|\lambda_{n-1} a_{n-1,n} b_n^{-1} \lambda_n^{-1}\right|\right| \cdot \left|\left|\lambda_n c_{n,n-1} b_{n-1}^{-1} \lambda_{n-1}^{-1}\right|\right|, \qquad n = 2, \ldots, N,$$

and let there be numbers $q_1, q_2, \ldots, q_N > 1$ with

$$\alpha_n \leq \frac{q_n - 1}{q_{n-1} q_n}, \qquad n \geq 2.$$

Then $\mathbf{H}_{0,N}x = 0$ and $\psi\mathbf{H} = 0$ can be solved by BackLU.

Concluding remarks:

- Often, LU decomposition refers to solving systems $Ax = b$ with some (square) invertible matrix A. Here, we consider a slight generalization: After performing the LU decomposition or the Gaussian elimination, the first row tells us whether we have no solution, exactly one solution or infinitely many solutions.
- Again, we can interprete $\mathbf{H}_{0,N}x = 0$ as an inhomogeneous system by fixing $x_0 = 1$. The inhomogeneous system has the form $\mathbf{H}_{1,N}y = b$. Under the conditions of our criteria, this system has a unique solution which can be obtained by BackLU.
- By renumbering, we could easily obtain criteria for the usual (forward) LU decomposition to work without pivoting.
- Taking the results of section 4.9 into account, we can state that our criteria still hold if $a_{mn} \in \mathbb{C}^{d_m \times d_n}$, $b_n \in \mathbb{C}^{d_n \times d_n}$, $c_{nm} \in \mathbb{C}^{d_n \times d_m}$. Hence, we have proved criteria for LU decomposition (without pivoting) to work for block-matrices.
- Such criteria can be found in the literature. For example, the block-tridiagonal case is considered in [GL96, section 4.5]. A typical assumption is strict block-diagonal dominance. Pringsheim-type criteria allow weak block-diagonal dominance, but it is important that $a_{n,n+1} \neq 0$ (or $c_{n+1,n} \neq 0$) for all $n = 0, \ldots, N-1$.
- The combination of weak diagonal dominance and some 'communication' assumption is far from new: For finite matrices, the term 'irreducible diagonally dominant' is standard, and used for stating simple criteria concerning the convergence of iterative methods for systems of linear equations, see e.g. [Saa03, Theorem 4.9]. Furthermore, for finite state spaces, it is well-known that the system $\psi\mathbf{P} = \psi$ for determining the invariant distribution of a discrete-time Markov chain can be solved without pivoting, see [Ste09, page 293]. Since the S-series interpretation is inspired by Markov chains, it does not surprise that the S-series criterion (and hence, Pringsheim-type criteria) generalize this result.

4.11 Literature review

There is few literature dealing with the topic of solving infinite systems of linear equations on Banach algebras by means of gcfs.

As pointed out in section 4.10, for finite systems, the gcf-approach is equivalent to a variant of LU decomposition or Gaussian elimination, and in some literature, criteria for LU decomposition to work without pivoting can be found, see [GL96] for example. These criteria usually refer to the case of block-matrices where the blocks are matrices with entries in $\mathbb{C}$.

For upper Hessenberg matrices, the system $\mathbf{H}x = 0$ reduces to a sum equation (with some initial condition). In the case $\mathbb{R} = \mathbb{C}$, there is some literature, we will discuss this topic in chapter 7 and in chapter 8.

As pointed out in section 4.10, it is well-known that for irreducible Markov chains with finite state spaces, the invariant distribution can be computed by LU decomposition without

pivoting, see [Ste09]. In total, many of our results directly apply to Markov chains, and we will discuss this application extensively in chapter 5. In the context of Markov chains, many special cases of our results are used in the literature, although quite few works explicitly use the term 'continued fraction'. We will refer to this literature at the end of chapter 5.

For stochastic matrices which are given as block matrices, Grassmann and Heyman [GH90] proved that block-Gaussian elimination is possible without pivoting. In fact, the approximants $v_n^{(N)} = v_0 R_{0,1,n}^{(N)}$ for v_n exactly correspond to π_n, where π_n is a solution of $\pi_n = \sum_{m=0}^{N} \pi_m p_{mn}$ with blocks p_{mn} for $m, n \in \{0, \ldots, N\}$. Hence, our definition of $v^{(N)}$ and $w^{(N)}$ respectively, and hence, the solution of finite systems of equations with block-matrices, can be interpreted as a straight-forward abstraction of the results in [GH90].

Chapter 5

Application to Markov chains

In section 2.1, we demonstrated the relationship between continued fractions and invariant measures for Markov chains with tridiagonal transition structure. More than that, we pointed out that our definition of **generalized** continued fractions is motivated by the wish of finding a gcf-based representation for invariant measures in case of arbitrary transition structures.

Actually, for irreducible Markov chains $X = (X_m)_{m\in\mathbb{N}_0}$ with **scalar** transition probability matrix $\mathbf{P} = (p_{ij})_{i,j\in\mathbb{N}_0}$, the basic results can be easily obtained: According to section 2.1 and section 2.3, we have

$$\begin{aligned}
K_n^{(N)} &= 1 - \mathbb{P}(\tau_n < T_{n,N} | X_0 = n), \\
K_n &= \lim_{N\to\infty} K_n^{(N)} = 1 - \mathbb{P}(\tau_n < T_n | X_0 = n), \\
\left(K_n^{(N)}\right)^{-1} &= 1 + \mathbb{E}\left[\sum_{m=1}^{T_{n,N}} \mathbf{1}_{\{n\}}(X_m) | X_0 = n\right], \\
R_{0,1,n}^{(N)} &= S(\mathbf{P}, 0, n, \mathbb{N}_{1,N}) = \mathbb{E}\left[\sum_{m=1}^{T_{1,N}} \mathbf{1}_{\{n\}}(X_m) | X_0 = n\right], \\
R_{0,1,n} &= \lim_{N\to\infty} R_{0,1,n}^{(N)} = \mathbb{E}\left[\sum_{m=1}^{\tau_0} \mathbf{1}_{\{n\}}(X_m) | X_0 = n\right].
\end{aligned}$$

The convergence conditions for the S-series are met (we will prove this claim in the more general context of Theorem 5.2.3), and hence, Theorem 4.7.1 directly yields that $v(v_0, \mathbf{H})$ solves $\psi\mathbf{P} = \psi$ (equivalent to $\psi\mathbf{H} = 0$) if $v_0 \mathrm{gcf}\,(\mathbf{H}) = 0$. In case of recurrence, we have $K_0 = 1 - \mathbb{P}(\tau_0 < \infty | X_0 = 0) = 1 - 1 = 0$, and hence this is true for any v_0. For positive v_0, we obtain an invariant measure. Instead of Theorem 4.7.1, we could refer to the fact that the interpretation of $R_{0,1,n}$ is well-known as $\frac{\psi_n}{\psi_0}$ for invariant measures ψ, see [Sen81].

Similarly, we have the interpretations

$$\begin{aligned}
L_{n,1,0}^{(N)} &= S(\mathbf{P}, n, 0, \mathbb{N}_{1,N}) = \mathbb{P}(\tau_0 < T_{1,N} | X_0 = n), \\
L_{n,1,0} &= \lim_{N\to\infty} L_{n,1,0}^{(N)} = \mathbb{P}(\tau_0 < \infty | X_0 = n),
\end{aligned}$$

and we directly obtain that $w_n(1, \mathbf{H}) = L_{n,1,0}$ is the probability of ever reaching state 0.

Alternatively, we could refer to the fact that these probabilities fulfill the system $\mathbf{H}x = 0$, and use Theorem 4.4.1.

Therefore, basic relationships between gcfs and invariant measures or absorption probabilities are a direct consequence of our construction of gcfs. Nevertheless, we will dedicate this chapter to a more detailed discussion. This has several reasons:

- As pointed out in the introduction, in many applications, we are not interested in the details of the invariant distribution ψ, but in computing ψf for a given function $f : E \to \mathbb{R}$ (where E is the state space). An approximation is given by

$$\psi^{(N)} f = \psi_0 \left(f(0) + \sum_{n=1}^{N} R^{(N)}_{0,1,n} f(n) \right) = K^{(N)}_{f,0},$$

  and according to (2.3.1), this is the approximant of a gcf K_f with $b_0 = f(0)$ and $c_{n0} = -f(n)$ for $n \geq 1$. In case of convergence, we obtain $\psi f = K_f$, and the speed-of-convergence results on K_f will yield upper bounds for $\psi f - \psi^{(N)} f$.

- Similarly, in many applications, we are not interested in computing the hitting (or absorption) probabilities $\mathbb{P}(\tau_0 < \infty | X_0 = n)$ for all initial states n. Instead, there is a given initial distribution $\alpha = (\alpha_n)_{n \in E}$, that is, $\alpha_n = \mathbb{P}(X_0 = n)$, and we are interested in the total probability of ever reaching state 0, that is,

$$\mathbb{P}(X_0 = 0 \text{ or } \tau_0 < \infty) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \mathbb{P}(\tau_0 < \infty | X_0 = n) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n L_{n,1,0}.$$

  Again, from (2.3.1), we obtain that we can write this total probability as a gcf by setting $b_0 = \alpha_0$ and $a_{0n} = \alpha_n$ for $n \geq 1$.

- As pointed out in section 1.3, in many applications, it makes sense to consider a partition $E = \dot{\bigcup}_{n \in \mathbb{N}_0} E_n$ of the state space, where the levels E_n can be finite, countably infinite or uncountably infinite. In the latter case, the one-step transition probabilities from states in level E_m to states in level E_n are characterized by kernels p_{mn}. For this reason, we will discuss kernels and an appropriate Banach algebra in section 5.1. Based on these considerations, in section 5.2, we will give gcf-based representations for absorption probabilities and invariant measures for discrete-time Markov processes with state spaces which are not necessarily discrete. Finally, we will use the concept of equivalence transformations to obtain similar results for continuous-time Markov chains (with discrete state space).

- If the E_n are finite, the representations of ψf and absorption probabilities can be exploited algorithmically. We will dedicate chapter 6 to presenting such methods.

5.1 Kernels

Since the p_{mn} shall characterize the one-step transition probabilities for transitions from level E_m to level E_n, they have to provide information on all transition probabilities $p_{mn}(x, A)$

from $x \in E_m$ to $A \in \mathcal{E}_n$, where $\mathcal{E}_n$ is an appropriate σ-algebra on $\mathcal{E}_n$. Such objects are known as transition kernels, and in the first subsection, we will start with defining a Banach algebra of transition kernels operating on a measurable space. These considerations would suffice if all E_n were isomorph. Since this assumption will not be met in all applications, in the second subsection, we will embed all E_n in one measurable space.

5.1.1 A Banach algebra of kernels

Let $(D, \mathcal{D})$ be a measurable space and let

- G be the Banach space of all measurable and bounded functions $g : D \to \mathbb{R}$, endowed with the supremum norm.
- F be the Banach space of all signed measures on $\mathcal{D}$ with finite total variation norm.

A signed measure is a function $\mu : \mathcal{D} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ (or $\mathbb{R}$, which does not matter for our purposes) with $\mu(\emptyset) = 0$ which is σ-additive. According to the Hahn decomposition, any signed measure μ can be written as $\mu^+ - \mu^-$, where both μ^+ and μ^- are unsigned measures. The total variation norm of an unsigned measure is given by

$$||\mu||_{TV} = |\mu|(D) = \mu^+(D) + \mu^-(D) = \sup_{g \in G:\ ||g|| \leq 1} \int \mu(dy) g(y).$$

For any Cauchy sequence in G or F respectively, it is easy to see that a limit exists in a pointwise sense. For proving that F or G respectively are Banach spaces, we need to show that the limits are still bounded and measurable (which is quite a popular result) or bounded signed measures respectively. For more details, we refer to [Die84].

(Signed) *transition kernels* are mappings $\kappa : D \times \mathcal{D} \to \mathbb{R}$, where $\kappa(\cdot, A)$ is a measurable function for all $A \in \mathcal{D}$, and $\kappa(x, \cdot)$ is a (signed) measure for all $x \in D$. We can introduce the norm

$$||\kappa|| = \sup_{x \in D} ||\kappa(x, \cdot)||_{TV},$$

and consider the set $\mathcal{R}$ of all signed transition kernels κ with $||\kappa|| < \infty$.

We define $\alpha\kappa_1 + \beta\kappa_2$ for $\alpha, \beta \in \mathbb{R}$ and $\kappa_1, \kappa_2 \in \mathcal{R}$ in the natural way, that is

$$(\alpha\kappa_1 + \beta\kappa_2)(x, A) = \alpha \cdot \kappa_1(x, A) + \beta \cdot \kappa_2(x, A),$$

and obtain that $\mathcal{R}$ is a normed vector space. Furthermore, if (κ_n) is a Cauchy sequence in $\mathcal{R}$, we can define a pointwise limit κ again. Since $||\kappa_n|| < \infty$ implies that $(\kappa_n(\cdot, A))$ is a G-valued sequence for all $A \in \mathcal{D}$, and $(\kappa_n(x, \cdot))$ is an F-valued sequence for all $x \in D$, and since G and F are Banach spaces, $\kappa(\cdot, A)$ is measurable and bounded, $\kappa(x, \cdot)$ is a signed measure with finite total variation, and hence, $\mathcal{R}$ is a Banach space.

Finally, we can introduce the multiplication $\kappa_1\kappa_2$ where

$$\kappa_1\kappa_2(x, A) = \int \kappa_1(x, dy)\kappa_2(y, A), \qquad x \in D,\ A \in \mathcal{D}.$$

Due to boundedness, the product is associative, and $||\kappa_1\kappa_2|| \leq ||\kappa_1|| \cdot ||\kappa_2||$ can be proved easily; the identity is given by $\mathrm{id}(x, A) = \delta_x(A)$, where δ_x is the Dirac measure in x. In total, $\mathcal{R}$ is a Banach algebra.

In a natural way, the kernels $\kappa \in \mathcal{R}$ define linear operators on G and F respectively.

- For $g \in G$ and $\kappa \in \mathcal{R}$, we can define $\kappa g \in G$ by

$$\kappa g(x) = \int \kappa(x, dy) g(y), \qquad x \in D.$$

  By choice of the norms, we have $||\kappa g|| \leq ||\kappa|| \cdot ||g||$.

- For $\mu \in F$ and $\kappa \in \mathcal{R}$, we can define $\mu\kappa \in F$ by

$$\mu\kappa(A) = \int \mu(dx)\kappa(x, A), \qquad A \in \mathcal{D}.$$

  Again, we directly obtain $||\mu\kappa|| \leq ||\mu|| \cdot ||\kappa||$.

In other words, we may interprete $\mathcal{R}$ as a sub-Banach algebra of $L(G)$ or $L(F)$ respectively. In both Banach spaces G and F, we have a natural definition of positivity. Hence, using the construction from appendix B.3, we can identify positive operators in $L(G)$ and $L(F)$ respectively. We will briefly demonstrate that

$$\mathcal{R}_+ = \{\kappa \in \mathcal{R} : \ \kappa(x, A) \geq 0 \text{ for all } x \in D, \ A \in \mathcal{D}\}$$

characterizes all positive operators in $\mathcal{R}$, that is

$$\mathcal{R}_+ = \mathcal{R} \cap (L(G))_+ = \mathcal{R} \cap (L(F))_+.$$

- For $g \in G$, we have $g \geq 0$ if $g(x) \geq 0$ for all $x \in D$. A linear operator $\in L(G)$ is positive if each positive g is mapped to a positive element. Obviously, we have $\kappa g \geq 0$ for all $\kappa \in \mathcal{R}_+$ and $g \geq 0$. On the other hand, for $A \in \mathcal{D}$ consider $g = \mathbf{1}_A \in G$. Trivially, we have $g = \mathbf{1}_A \geq 0$, and hence $\kappa \in \mathcal{R} \cap (L(G))_+$ requires $\kappa\mathbf{1}_A \geq 0$, that is

$$\kappa\mathbf{1}_A(x) = \int \kappa(x, dy)\mathbf{1}_A(y) = \kappa(x, A) \geq 0$$

  for all $x \in D$.

- For $\mu \in F$, we have $\mu \geq 0$ if μ is an unsigned measure, that is $\mu(A) \geq 0$ for all $A \in \mathcal{D}$. Again, we obviously have $\mu\kappa \geq 0$ for all $\mu \geq 0$ and $\kappa \in \mathcal{R}_+$. On the other hand, we have $\delta_x \geq 0$ for the Dirac-measure in any $x \in D$. Hence, $\kappa \in \mathcal{R} \cap (L(F))_+$ requires $\delta_x\kappa \geq 0$, that is

$$\delta_x\kappa(A) = \int \delta_x(dy)\kappa(y, A) = \kappa(x, A) \geq 0$$

  for all $A \in \mathcal{D}$.

The elements of $\mathcal{R}^+$ are *unsigned bounded kernels*, and in the Markov-chain context, the kernels p_{mn} characterizing the one-step transition probabilities from level E_m to level E_n are indeed elements of $\mathcal{R}_+$ with $||p_{mn}|| \leq 1$ if all E_n are isomorphic to D.

Some concluding remarks:

- Note that we did not state $\mathcal{R} = L(G)$ or $\mathcal{R} = L(F)$ (which would not be true), but only $\mathcal{R} \subset L(G)$ and $\mathcal{R} \subset L(F)$. Nevertheless, since $\mathcal{R}_+ = \mathcal{R} \cap (L(G))_+$, the results in Theorem B.3.1 directly apply to $\mathcal{R}_+$, that is, a convergent series $\sum_{n=1}^{\infty} \kappa_n$ with $\kappa_n \in \mathcal{R}_+$ converges unconditionally.
- For our purposes, we do not need any further considerations of the structure of $L(G)$ and $L(F)$. For details on this topic, we refer to [Die84] again.
- Signed kernels κ can be written as $\kappa = \kappa^+ - \kappa^-$ with $\kappa^+(x, A), \kappa^-(x, A) \geq 0$. If $\mathcal{E}$ is countably generated (e.g. $(E, \mathcal{E})$ is a Polish space), it turns out that κ^+ and κ^- are unsigned kernels (measurability in the first component is a consequence of [DF64, Theorem 2.8]). For further issues on unsigned kernels, we refer to [Kle06; MT93].

5.1.2 Modifications for different E_n

As pointed out above, we will now consider the case where we have different measurable spaces $(E_1, \mathcal{E}_1)$, $(E_2, \mathcal{E}_2)$, ... Then the kernel characterizing transition probabilities from states in E_m to states in E_n is a function $E_m \times \mathcal{E}_m \to \overline{\mathbb{R}}$ which is measurable in the first component and which is a (signed) measure in the second one. Now assume that

- there is a Polish space $(E, \mathcal{E})$ such that $E_0, E_1, \ldots \in \mathcal{E}$, yielding Polish spaces $(E_n, \mathcal{E}_n)$ with σ-algebra $\mathcal{E}_n = \{B \cap E_n : \ B \in \mathcal{E}\}$.
- for $m, n \in \mathbb{N}_0$, $p_{mn} : E_m \times \mathcal{E}_n \to \overline{\mathbb{R}}$ is a (non-signed) kernel with finite operator norm, that is, $x \mapsto p_{mn}(x, B)$ is measurable, $B \mapsto p_{mn}(x, B)$ is a finite measure and $||p_{mn}|| = \sup_{x \in E_m} p_{mn}(x, E_n) < \infty$.

In this situation, we can embed the p_{mn} in the algebra $\mathcal{R}$ of kernels on $E \times \mathcal{E}$, using a similar idea as in section 4.9: Define $\tilde{p}_{mn}$ by

$$\tilde{p}_{mn}(x, B) = \begin{cases} p_{mn}(x, B \cap E_n), & x \in E_m, \ B \in \mathcal{E}, \\ 0, & x \in E \setminus E_m, \ B \in \mathcal{E}. \end{cases}$$

Simple induction yields

$$\left(\prod_{r=1}^{n} \tilde{p}_{i_{r-1}, i_r} \right)(x, B) = \begin{cases} \left(\prod_{r=1}^{n} p_{i_{r-1}, i_r} \right)(x, B \cap E_{i_n}), & x \in E_{i_0}, \\ 0, & x \in E \setminus E_{i_0}, \end{cases}$$

and hence,

$$S\left(\tilde{\mathbf{P}}, m, n, A\right)(x, B) = \begin{cases} S(\mathbf{P}, m, n, A)(x, B \cap E_n), & x \in E_m, \\ 0, & x \in E \setminus E_m. \end{cases}$$

These considerations enable us to apply our former results.

Theorem 5.1.1. *Let $(E, \mathcal{E})$ be a Polish space, let*

- *$E_n \in \mathcal{E}$ for all $n \in \mathbb{N}_0$, $\mathcal{E}_n = \{B \cap E_n : \ B \in \mathcal{E}\}$,*

- *b_n be a (signed) kernel on $E_n \times \mathcal{E}_n$, a_{mn} be a (signed) kernel on $E_m \times \mathcal{E}_n$, c_{nm} be a (signed) kernel on $E_n \times \mathcal{E}_m$ for all $m, n \in \mathbb{N}_0$, $m < n$,*
- *λ_n and ρ_n be invertible kernels on $E_n \times \mathcal{E}_n$ for all $n \in \mathbb{N}_0$,*
- *$p_{nn} = \mathrm{id}_{E_n} - \lambda_n b_n \rho_n^{-1}$ (where id_{E_n} is the identity kernel on $E_n \times \mathcal{E}_n$) for all $n \in \mathbb{N}_0$, $p_{mn} = -\lambda_m a_{mn} \rho_n^{-1}$, $p_{nm} = \lambda_n c_{nm} \rho_m^{-1}$ for all $m, n \in \mathbb{N}_0$, $m < n$,*
- *$p_{mn} \geq 0$ for all $m, n \in \mathbb{N}_0$, that is $p_{mn}(x, \cdot)$ is a (non-signed) measure on $\mathcal{D}_n$ for all $x \in E_m$,*
- *$||p_{mn}|| < \infty$ for all $m, n \in \mathbb{N}_0$,*
- *$S(\mathbf{P}, 0, 0, \mathbb{N})$ converge,*
- *$S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ converge for all $n, N \in \mathbb{N}$, $n \leq N$, and*
- *$S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ converge for all $k, n, m, N \in \mathbb{N}_0$ with $k < n \leq m \leq N$ or let $S(\mathbf{P}, k, m, \mathbb{N}_{n,N})$ converge for all $k, n, m, N \in \mathbb{N}_0$ with $k < n \leq m \leq N$.*

Then $\mathrm{gcf}\,(\mathbf{H})$ *converges with* $\mathrm{gcf}\,(\mathbf{H}) = \lambda_0^{-1}(\mathrm{id}_{E_0} - S(\mathbf{P}, 0, 0, \mathbb{N}))\rho_0$. *Furthermore, if there is a sequence*

a) $(g_n)_{n \in \mathbb{N}_0}$ *of measurable and bounded functions* $g_n : E_n \to \mathbb{R}$ *such that*

$$\sum_{k=0}^{\infty} p_{mk} g_k \leq g_m, \qquad m \in \mathbb{N},$$

we have

$$0 \leq \lambda_0 \left(K_0^{(N)} - \mathrm{gcf}\,(\mathbf{H}) \right) \rho_0^{-1} g_0 \leq \sum_{m=N+1}^{\infty} S(\mathbf{P}, 0, m, \mathbb{N}_{1,N}) g_m.$$

b) $(\mu_n)_{n \in \mathbb{N}_0}$ *of finite measures* μ_n *on* $\mathcal{E}_n$ *such that*

$$\sum_{k=0}^{\infty} \mu_k p_{km} \leq \mu_m, \qquad m \in \mathbb{N},$$

we have

$$0 \leq \mu_0 \lambda_0 \left(K_0^{(N)} - \mathrm{gcf}\,(\mathbf{H}) \right) \rho_0^{-1} \leq \sum_{m=N+1}^{\infty} \mu_m S(\mathbf{P}, m, 0, \mathbb{N}_{1,N}).$$

Proof. The statements follow directly from our above considerations and Theorem 4.8.2, where for the second inequality, we have to consider the 'transposed' system due to p_{mn} operating from the right instead of operating from the left. The kernels b_n, a_{mn}, c_{nm} are embedded in $\mathcal{R}$ with the same construction as p_{mn}, for the transformation kernels λ_n, we define $\tilde{\lambda}_n : E \times \mathcal{E} \to \mathbb{R} \cup \{-\infty, \infty\}$ by $\tilde{\lambda}_n(x, B) = \lambda_n(x, B \cap E_n)$ for $x \in E_n$, and $\tilde{\lambda}_n(x, B) = \mathbf{1}_B(x)$ for $x \notin E_n$; and analogously, we define $\tilde{\rho}_n$. Then we still have $\tilde{p}_{nn} = I - \tilde{\lambda}_n \tilde{b}_n \tilde{\rho}_n^{-1}$ with the identity kernel I on $E \times \mathcal{E}$, and similar relationships hold for $\tilde{a}_{mn}$ and $\tilde{c}_{nm}$. Furthermore, the definition of $\tilde{\lambda}_n$ and $\tilde{\rho}_n$ preserve the invertibility. □

Theorem 5.1.2. *Let $(E, \mathcal{E})$ be a Polish space, let*

- *$E_n \in \mathcal{D}$ for all $n \in \mathbb{N}_0$, $\mathcal{E}_n = \{B \cap E_n : \ B \in \mathcal{E}\}$,*
- *b_n be a kernel on $E_n \times \mathcal{E}_n$, a_{mn} be a kernel on $E_m \times \mathcal{E}_n$, c_{nm} be a kernel on $E_n \times \mathcal{E}_m$ for all $m, n \in \mathbb{N}_0$, $m < n$,*
- *$\lambda_n \geq 0$ and $\rho_n \geq 0$ be invertible kernels on $E_n \times \mathcal{E}_n$ for all $n \in \mathbb{N}_0$,*
- *$p_{nn} = \mathrm{id}_{E_n} - \lambda_n b_n \rho_n^{-1}$ (where id_{E_n} is the identity kernel on $E_n \times \mathcal{E}_n$) for all $n \in \mathbb{N}_0$, $p_{mn} = -\lambda_m a_{mn} \rho_n^{-1}$, $p_{nm} = \lambda_n c_{nm} \rho_m^{-1}$ for all $m, n \in \mathbb{N}_0$, $m < n$,*
- *$p_{mn} \geq 0$ for all $m, n \in \mathbb{N}_0$, that is $p_{mn}(x, \cdot)$ is a measure on $\mathcal{E}_n$ for all $x \in E_m$,*
- *$||p_{mn}|| < \infty$ for all $m, n \in \mathbb{N}_0$,*
- *$S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$ converge for all $n, N \in \mathbb{N}$, $n \leq N$.*

Furthermore, let

a) $S(\mathbf{P}, n, 0, \mathbb{N})$ converge for all $n \in \mathbb{N}_0$, and let $S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ converge for all $k, n, m, N \in \mathbb{N}_0$ with $k < n \leq m \leq N$. Then $\mathrm{gcf}\,(\mathbf{H})$ and $w = (w_n) = w(w_0, \mathbf{H})$ converge for all kernels w_0 on $E_0 \times \mathcal{E}_0$ with $||w_0|| < \infty$. w_n is a kernel on $E_n \times \mathcal{E}_n$ with $||w_n|| < \infty$, and w satisfies (4.1.2), that is,

$$\sum_{m=0}^{n-1} c_{nm} w_m = b_n w_n + \sum_{m=n+1}^{\infty} a_{nm} w_m, \qquad n \in \mathbb{N},$$

and for any sequence $(x_n)_{n \in \mathbb{N}_0}$ of kernels x_n on $E_n \times \mathcal{E}_n$ with $||x_n|| < \infty$, $x_0 \geq w_0$ and

$$\sum_{m=0}^{n-1} c_{nm} x_m \geq b_n x_n + \sum_{m=n+1}^{\infty} a_{nm} x_m, \qquad n \in \mathbb{N},$$

we have $x_n \geq w_n$ for all $n \in \mathbb{N}_0$. If additionally $\mathrm{gcf}\,(\mathbf{H})\, w_0 = 0$, we have $\mathbf{H} w = 0$.

b) $S(\mathbf{P}, 0, n, \mathbb{N})$ converge for all $n \in \mathbb{N}_0$, and let $S(\mathbf{P}, k, m, \mathbb{N}_{n,N})$ converge for all $k, n, m, N \in \mathbb{N}_0$ with $k < n \leq m \leq N$. Then $\mathrm{gcf}\,(\mathbf{H})$ and $v = (v_n) = v(v_0, \mathbf{H})$ converge for all kernels v_0 on $E_0 \times \mathcal{E}_0$ with $||v_0|| < \infty$. v_n is a kernel on $E_n \times \mathcal{E}_n$ with $||v_n|| < \infty$, and v satisfies (4.7.2), that is,

$$\sum_{m=n+1}^{\infty} v_m c_{mn} = \sum_{m=0}^{n-1} v_m a_{mn} + v_n b_n, \qquad n \in \mathbb{N},$$

and for any sequence $(\psi_n)_{n \in \mathbb{N}_0}$ of kernels ψ_n on $E_n \times \mathcal{E}_n$ with $||\psi_n|| < \infty$, $\psi_0 \geq v_0$ and

$$\sum_{m=n+1}^{\infty} \psi_m c_{mn} \geq \sum_{m=0}^{n-1} \psi_m a_{mn} + \psi_n b_n, \qquad n \in \mathbb{N},$$

we have $\psi_n \geq v_n$ for all $n \in \mathbb{N}_0$. If additionally $v_0 \mathrm{gcf}\,(\mathbf{H}) = 0$, we have $v\mathbf{H} = 0$.

Proof. Direct consequence of our considerations concerning kernels and Theorem 4.8.1. □

Remark 5.1.1. The construction of $E_n \subset E$ corresponds to allowing block matrices of various dimensions in section 4.9: Put $E = \mathbb{N}$ and $E_n = \{1, \ldots, d_n\}$.

5.2 Application to discrete-time Markov processes

In the literature, the usage of the term 'Markov chain' is not unique: Sometimes it refers to Markov processes with discrete state space, sometimes it refers to Markov processes in discrete time, and sometimes it refers to Markov processes with discrete time and discrete state space. Here, we will consider all of these Markov processes, and we will use the terms

- DTMP for discrete-time Markov processes, that is, for Markov processes in discrete time with general state space.
- DTMC for discrete-time Markov chains, that is, for Markov processes in discrete time with discrete state space.
- CTMC for continuous-time Markov chains, that is, for Markov processes in continuous time with discrete state space.

We refer to Appendix C for more details. We begin with considering a DTMP $X = (X_m)_{m\in\mathbb{N}_0}$ with Polish state space $(E, \mathcal{E})$ with $E = \dot{\bigcup\limits_{n\in\mathbb{N}_0}} E_n$, where $E_n \in \mathcal{E}$ for all $n \in \mathbb{N}$. Let $\mathcal{E}_n = \{B \cap E_n : \ B \in \mathcal{E}\}$, and let p_{mn} denote the sub-Markovian kernel with

$$p_{mn}(x, B) = \mathbb{P}_x(X_1 \in B), \qquad x \in E_m, \ B \in \mathcal{E}_n.$$

In case $\mathbb{P}(X_0 = x) > 0$, this means $p_{mn}(x, B) = \mathbb{P}(X_1 \in B|X_0 = x)$, see appendix C for more details on $\mathbb{P}_x$.

Of course, we have $p_{mn} \geq 0$ and $||p_{mn}|| \leq 1 < \infty$. As in Appendix C, we define

$$\sigma_B = \inf\{m \in \mathbb{N}_0 : \ X_m \in B\}, \qquad \tau_B = \inf\{m \in \mathbb{N} : \ X_m \in B\} \text{ and } \qquad \eta_B = \sum_{m=1}^{\infty} \mathbf{1}_B(X_m)$$

for $B \in \mathcal{E}$. In case of $\mathbb{P}_x(\tau_B < \infty) > 0$, we write $x \to B$. In particular, we will use σ_{E_0}, τ_{E_0} and the condition $\forall x \in E \setminus E_0 : \ x \to E_0$.

When applying Theorem 5.1.1 or Theorem 5.1.2 – up to a few easy exceptions – we directly use $b_n = I - p_{nn}$, $a_{mn} = -p_{mn}$ and $c_{nm} = p_{nm}$, that is, we need no equivalence transformation (and thus, there is no need to make explicit use of b_n, a_{mn} and c_{nm}). Then the S-series still have a probabilistic interpretation: The Markov property directly yields

$$\left(\prod_{r=1}^{n} p_{i_{r-1},i_r}\right)(x, B) = \mathbb{P}_x\left(X_{i_1} \in E_1, \ldots, \ X_{i_{n-1}} \in E_{n-1}, \ X_{i_n} \in B\right), \ x \in E_{i_0}, \ B \in \mathcal{E}_{i_n},$$

and hence

$$S(\mathbf{P}, m, n, A)(x, B) = \sum_{\ell=1}^{\infty} \mathbb{P}_x\left(X_\ell \in B, \ X_1, \ldots, X_{\ell-1} \in \bigcup_{i\in A} E_i\right), \ x \in E_m, \ B \in \mathcal{E}_n.$$

Important special cases are

- the hitting probabilities

$$\begin{aligned} S(\mathbf{P}, m, n, \mathbb{N}_0 \setminus \{n\})(x, B) &= \sum_{\ell=1}^{\infty} \mathbb{P}_x(X_\ell \in B,\ X_1, \dots, X_{\ell-1} \notin E_n) \\ &= \mathbb{P}_x(\tau_{E_n} < \infty,\ X_{\tau_{E_n}} \in B), \quad x \in E_m,\ B \in \mathcal{E}_n. \end{aligned}$$

  Obviously, $S(\mathbf{P}, m, n, \mathbb{N}_0 \setminus \{n\})(x, E_n) \leq 1$, that is, $S(\mathbf{P}, m, n, \mathbb{N}_0 \setminus \{n\})$ converges and is sub-Markovian.

- the mean occupation times

$$S(\mathbf{P}, m, n, \mathbb{N}_0)(x, B) = \sum_{\ell=1}^{\infty} \mathbb{P}_x(X_\ell \in B) = \mathbb{E}_x\left[\eta_B\right], \qquad x \in E_m,\ B \in \mathcal{E}_n.$$

  In particular, $S(\mathbf{P}, m, n, \mathbb{N}_0)(x, E_n) = \mathbb{E}_x\left[\eta_{E_n}\right]$. For convergence, this expectation has to be bounded in x. A sufficient condition is given by uniform transience of E_n (see Appendix C).

In our results, we will always assume that $E_1, E_2, \dots$ are petite (see Appendix C). If E_0 is absorbing, that is, p_{00} is a Markov kernel, and if $x \to E_0$ for all $x \notin E_0$, petiteness of E_n yields uniform transience of E_n (see Lemma C.2.1).

With these preparations, we are able to state gcf-representations for absorption probabilities, mean absorbing times, and invariant measures, which can be algorithmically exploited for finite sets $E_0, E_1, E_2, \dots$.

Theorem 5.2.1. *Let (X_m) be a DTMP with Polish state space $(E, \mathcal{E})$ with $E = \dot{\bigcup_{n \in \mathbb{N}_0}} E_n$, where $E_n \in \mathcal{E}$ for all $n \in \mathbb{N}_0$. For $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, let $p_{mn}(x, B) = \mathbb{P}_x(X_1 \in B)$ for $x \in E_m$, $B \in \mathcal{E}_n$ and $\alpha_n(B) = \mathbb{P}(X_0 \in B)$ for $B \in \mathcal{E}_n$. Let E_n be petite for $n \geq 1$, and let $x \to E_0$ for all $x \in E \setminus E_0$. Finally, let*

$$\mathbf{H} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ -p_{10} & \mathrm{id}_{E_1} - p_{11} & -p_{12} & \dots \\ -p_{20} & -p_{21} & \mathrm{id}_{E_2} - p_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

a) $w(I, \mathbf{H})$ converges with the interpretation

$$w_n(I, \mathbf{H})(x, B) = \mathbb{P}_x(\sigma_{E_0} < \infty, X_{\sigma_{E_0}} \in B), \qquad x \in E_n,\ B \in \mathcal{E}_0.$$

b) $\mathrm{gcf}\,(\mathbf{H})$ converges as a measure on $\mathcal{E}_0$ with the interpretation

$$\mathrm{gcf}\,(\mathbf{H})\,(B) = \mathbb{P}(\sigma_{E_0} < \infty, X_{\sigma_{E_0}} \in B), \qquad B \in \mathcal{E}_0,$$

c) If there is a sequence $(\mu_n)_{n \in \mathbb{N}}$ of finite measures μ_n on $\mathcal{E}_n$ with

$$\alpha_m + \sum_{k=1}^{\infty} \mu_k p_{km} \leq \mu_m, \qquad m \in \mathbb{N},$$

we have

$$0 \leq \mathrm{gcf}\,(\mathbf{H}) - K_0^{(N)} \leq \sum_{m=N+1}^{\infty} \mu_m p_{m0} + \sum_{n=1}^{N} \sum_{m=N+1}^{\infty} \mu_m p_{mn} w_n^{(N)}.$$

d) Let $(F,\mathcal{F})$ be another Polish space, let $h \geq 0$ be a kernel on $E_0 \times \mathcal{F}$ with $||h|| < \infty$ and replace p_{n0} by $p_{n0}h$ for $n > 0$ and and α_0 by $\alpha_0 h$. Then the convergence and speed-of-convergence statements in a), b) and c) still hold, and we have the interpretations

$$\begin{aligned} w_n(I,\mathbf{H})(x,B) &= \int \mathbb{P}_x(\sigma_{E_0} < \infty, X_{\sigma_{E_0}} \in dy) h(y,B), \qquad x \in E_n,\ B \in \mathcal{F}, \\ \operatorname{gcf}(\mathbf{H})(B) &= \int \mathbb{P}(\sigma_{E_0} < \infty, X_{\sigma_{E_0}} \in dy) h(y,B), \qquad B \in \mathcal{F}. \end{aligned}$$

Proof. In principle, we apply Theorem 5.1.1 and Theorem 5.1.2, where we can directly use our transition probabilities p_{mn} for $m \geq 1$. The transition probabilities $\mathbb{P}_x(X_1 \in B)$ for $x \in E_0$ have no impact on the hitting probabilities, and we will not use p_{0n} in our proof.

a) First, we have to prove the convergence of some S-series, using our above interpretations.

- For all $n \in \mathbb{N}$, $S(\mathbf{P}, n, 0, \mathbb{N})$ is the hitting probability for E_0, and as a probability, it converges.
- Next, we consider $S(\mathbf{P}, n, n, \mathbb{N}_{n,N}) = S\left(\tilde{\mathbf{P}}, n, n, \mathbb{N}_{n,N}\right) \leq S\left(\tilde{\mathbf{P}}, n, n, \mathbb{N}_0\right)$, where $\tilde{p}_{0n} = \delta_{n0} \operatorname{id}_{E_0}$ and $\tilde{p}_{mn} = p_{mn}$ for $m \geq 1$. For $\tilde{\mathbf{P}}$, E_0 is absorbing, and petiteness of D_n and $x \to E_0$ for all $x \in E_n$ yield that E_n is uniformly transient for $n \geq 1$ due to Lemma C.2.1. The interpretation of $S\left(\tilde{\mathbf{P}}, n, n, \mathbb{N}_0\right)$ as expected occupation time yields its convergence.
- For $k < n$, $S(\mathbf{P}, m, k, \mathbb{N}_{n,N})$ is a hitting probability, and hence, converges.

Due to $w_n = L_{n,1,0} = S(\mathbf{P}, n, 0, \mathbb{N})$, we can directly state the interpretation of w_n. Alternatively, we define the kernel $G : E \times \mathcal{E}_0$ as in Theorem C.2.2 (with $C = E_0$), that is,

$$G(x,B) = \mathbb{P}_x(\sigma_{E_0} < \infty, X_{\sigma_{E_0}} \in B), \qquad x \in E,\ B \in \mathcal{E}_0$$

and addditionally, we define the restrictions $G_n(x,B) : E_n \times \mathcal{E}_0$. Then Theorem C.2.2 states that we have $G_0 = \operatorname{id}_{E_0}$, $G_m = \sum_{m=0}^{\infty} p_{mn} G_n$ for $m \geq 1$, and for any other sequence (H_n) of kernels H_n on $E_n \times \mathcal{E}_0$ with $H_0 \geq \operatorname{id}_{E_0}$ and $H_m \geq \sum_{m=0}^{\infty} p_{mn} H_n$ for $m \geq 1$, we have $H_m \geq G_m$. Since Theorem 5.1.2 states that the gcf-based solution (w_n) is minimal, we obtain the statement again.

b) Formally, we consider

$$\tilde{\mathbf{H}} = \begin{pmatrix} I_{E_0} - \alpha_0 & -\alpha_1 & -\alpha_2 & \dots \\ -p_{10} & \operatorname{id}_{E_1} - p_{11} & -p_{12} & \dots \\ -p_{20} & -p_{21} & \operatorname{id}_{E_2} - p_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad \text{and}$$

$$\tilde{\mathbf{P}} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ p_{10} & p_{11} & p_{12} & \dots \\ p_{20} & p_{21} & p_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Then $\operatorname{gcf}(\mathbf{H}) = I_{E_0} - \operatorname{gcf}\left(\tilde{\mathbf{H}}\right) = S\left(\tilde{\mathbf{P}}, 0, 0, \mathbb{N}\right)$, and for convergence, it suffices to prove convergence of $S\left(\tilde{\mathbf{P}}, 0, 0, \mathbb{N}\right)$, which is done by the interpretation in the statement. *Remark:* Precisely, we have to put $\tilde{p}_{0n}(x, B) = \alpha_0(B)$ in order to obtain kernels. Since the result does not depend on x, we may multiply by δ_x for some $x \in E_0$ afterwards, and get back to measures.

c) The speed-of-convergence statement in Theorem 5.1.1 yields

$$\operatorname{gcf}(\mathbf{H}) - K_0^{(N)} = \tilde{K}_0^{(N)} - \operatorname{gcf}\left(\tilde{\mathbf{H}}\right) \leq \sum_{m=N+1}^{\infty} \mu_m S\left(\tilde{\mathbf{P}}, m, 0, \mathbb{N}\right)$$

if $\sum_{k=0}^{\infty} \mu_k \tilde{p}_{km} \leq \mu_m$ for all $m \in \mathbb{N}$. The sequence (μ_n) can be multiplied by a constant factor, and hence, we can choose μ_0 as a probability measure. Then $\mu_0 \tilde{p}_{0m}(B) = \int \mu_0(dx)\alpha_m(B) = \alpha_m(B)$, and due to $\tilde{p}_{km} = p_{km}$ for $k \geq 1$, the condition becomes equivalent to $\alpha_m + \sum_{k=1}^{\infty} \mu_k p_{km} \leq \mu_m$. The statement then follows from

$$\begin{aligned} S\left(\tilde{\mathbf{P}}, m, 0, \mathbb{N}_{1,N}\right) &= S(\mathbf{P}, m, 0, \mathbb{N}_{1,N}) = p_{m0} + \sum_{n=1}^{N} p_{mn} S(\mathbf{P}, n, 0, \mathbb{N}_{1,N}) \\ &= p_{m0} + \sum_{n=1}^{N} p_{mn} w_n^{(N)}. \end{aligned}$$

d) The multiplication of p_{n0} and α_0 with the bounded and fixed kernel h has no influence on the convergence properties and the interpretations are clear from multiplying with h.

□

The factor h can make sense in some situations, in particular when algorithmically computing hitting (or absorption) probabilities. Then E_0 can be infinitely large, and the kernel h may reduce the problem to a computation of finitely many values. As an example, let $E_0 = [0, 1]$, and let $h(x, \{1\}) = 1$ and $h(x, \{2\}) = x$. Then $\operatorname{gcf}(\mathbf{H})$ is a two-dimensional vector, the first component is the hitting probability on E_0, the second one is the moment of the state $\in E_0$ which the DTMP hits first.

Theorem 5.2.2. *Let (X_m) be a DTMP with state space $(E, \mathcal{E})$ with $E = \dot{\bigcup}_{n \in \mathbb{N}_0} E_n$, where $E_n \in \mathcal{E}$ for all $n \in \mathbb{N}_0$. For $m, n \in \mathbb{N}$ let $p_{mn}(x, B) = \mathbb{P}_x(X_1 \in B)$ for $x \in E_m$, $B \in \mathcal{E}_n$ and $\alpha_n(B) = \mathbb{P}(X_0 \in B)$ for $B \in \mathcal{E}_n$. Let E_n be petite for $n \geq 1$, and let $x \to E_0$ for all $x \in E \setminus E_0$. Finally, let*

$$\mathbf{H} = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \dots \\ -f_1 & \mathrm{id}_{E_1} - p_{11} & -p_{12} & \dots \\ -f_2 & -p_{21} & \mathrm{id}_{E_2} - p_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where $f_n : E_n \to \mathbb{R}_{\geq 0}$ is bounded and measurable.

a) We have

$$w_n(I,\mathbf{H})(x) = \mathbb{E}_x\left[\sum_{k=0}^{\sigma_{E_0}-1} f(X_k)\right], \qquad x \in E_n,\ n \in \mathbb{N},$$

where both sides take values in $[0,\infty]$.

b) We have

$$\operatorname{gcf}(\mathbf{H}) = \mathbb{E}\left[\sum_{k=0}^{\sigma_{E_0}-1} f(X_k)\right],$$

where both sides take values in $[0,\infty]$.

Proof. $\mathbb{E}[\ldots]$ is not influenced by p_{0n}, and the only information from p_{n0} we need is $p_{n0}(x,E_0)$. Hence, without restriction, we assume $|E_0| = 1$. Again, we write

$$\operatorname{gcf}(\mathbf{H}) = S\left(\tilde{\mathbf{P}}, 0, 0, \mathbb{N}\right),$$

where

$$\tilde{p}_{mn} = \begin{cases} p_{mn}, & m,n \in \mathbb{N}, \\ \alpha_n, & m = 0,\ n \geq 1, \\ f_m, & m \geq 1,\ n = 0, \\ 0, & m = n = 0. \end{cases}$$

Let $g(x) = \mathbb{E}_x\left[\sum_{k=0}^{\sigma_{E_0}-1} f(X_k)\right]$ and $g_n : D_n \to \mathbb{R}$, $g_n(x) = g(x)$. Then Theorem C.2.3 states that

$$g_m = \sum_{n=1}^{\infty} p_{mn} g_n + f_n, \qquad m \in \mathbb{N},$$

and $g_n \leq h_n$ for any sequence (h_n) of measurable functions on D_n with

$$h_m \geq \sum_{n=0}^{\infty} p_{mn} h_n + f_n, \qquad m \in \mathbb{N}.$$

Hence, Theorem 5.1.2 yields $w_n = g_n$ in case of convergence. If $g_n < \infty$, the minimality of (w_n) yields $w_n < \infty$, and if $w_n < \infty$, the minimality of (g_n) yields $g_n < \infty$. The statement for $\operatorname{gcf}(\mathbf{H})$ is a direct application of total probability. □

Theorem 5.2.3. *Let (X_m) be a DTMP with state space $(E,\mathcal{E})$ with $E = \dot{\bigcup}_{n\in\mathbb{N}_0} E_n$, where $E_n \in \mathcal{E}$ for all $n \in \mathbb{N}_0$. For $m,n \in \mathbb{N}_0$ let $p_{mn}(x,B) = \mathbb{P}_x(X_1 \in B)$ for $x \in E_m$, $B \in \mathcal{E}_n$.*

Let φ be an irreducibility measure with $\varphi(E_0) > 0$, let E_n be petite for all $n \in \mathbb{N}_0$, let E_0 be Harris recurrent, and let

$$\mathbf{H} = \begin{pmatrix} \mathrm{id}_{E_0} - p_{00} & -p_{01} & -p_{02} & \dots \\ -p_{10} & \mathrm{id}_{E_1} - p_{11} & -p_{12} & \dots \\ -p_{20} & -p_{21} & \mathrm{id}_{E_2} - p_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \qquad \textit{and}$$

$$\mathbf{H}_f = \begin{pmatrix} \psi_0 f_0 & -\psi_0 p_{01} & -\psi_0 p_{02} & \dots \\ f_1 & \mathrm{id}_{E_1} - p_{11} & -p_{12} & \dots \\ f_2 & -p_{21} & \mathrm{id}_{E_2} - p_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where ψ_0 is a finite measure on $\mathcal{E}_0$, (f_n) is a sequence of measurable and bounded functions $f_n : E_n \to \mathbb{R}_{\geq 0}$, and $f : E \to \mathbb{R}$ is defined by $f(x) = f_n(x)$ for $x \in E_n$.

a) $v(I, \mathbf{H})$ converges with the interpretation

$$v_n(I, \mathbf{H})(x, B) = \sum_{k=0}^{\infty} \mathbb{P}_x(\tau_{E_0} > k,\ X_k \in B), \qquad x \in E_0,\ B \in \mathcal{E}_n.$$

b) $\psi = v(\psi_0, \mathbf{H})$ converges for any finite measure ψ_0 on $\mathcal{E}_0$. Then, ψ_n is a finite measure on $\mathcal{E}_n$, and the sequence (ψ_n) satisfies

$$\sum_{k=0}^{\infty} \psi_k p_{km} = \psi_m, \qquad m \in \mathbb{N}.$$

c) $\mathrm{gcf}\,(\mathbf{H})$ converges and $T = \mathrm{id}_{E_0} - \mathrm{gcf}\,(\mathbf{H})$ is a $\varphi|_{E_0}$-irreducible and recurrent Markov kernel with the interpretation

$$T(x, B) = \mathbb{P}(\tau_{E_0} < \infty, X_{\tau_{E_0}} \in B | X_0 = x), \qquad x \in E_0,\ B \in \mathcal{E}_0.$$

Invariance of ψ_0 for T is equivalent to $\mathrm{gcf}\,(\mathbf{H}) = 0$ and implies invariance of ψ for $\mathbf{P}$.

d) For any finite measure ψ_0 on $\mathcal{E}_0$, we can interprete

$$\mathrm{gcf}\,(\mathbf{H}_f) = \int_E \psi(dx) f(x),$$

where $\psi = v(\psi_0, \mathbf{H})$.

e) If there is a sequence $(g_n)_{n \in \mathbb{N}}$ of bounded and measurable functions $g_n : E_n \to \mathbb{R}_{\geq 0}$ with

$$f_m + \sum_{k=1}^{\infty} p_{mk} g_k \leq g_m, \qquad m \in \mathbb{N},$$

we have

$$0 \leq \mathrm{gcf}\,(\mathbf{H}_f) - K_{f,0}^{(N)} \leq \sum_{m=N+1}^{\infty} \psi_0 p_{0m} g_m + \sum_{n=1}^{N} \psi_n^{(N)} \sum_{m=N+1}^{\infty} p_{nm} g_m,$$

where $K_{f,0}^{(N)}$ is the Nth approximant for $\mathrm{gcf}\,(\mathbf{H}_f)$ and $\psi_n^{(N)} = v_n^{(N)}(\psi_0, \mathbf{H})$.

Proof. a) For the proof of convergence of $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$, we refer to the proof of Theorem 5.2.1. Further, we write

$$\begin{aligned} S(\mathbf{P}, k, m, \mathbb{N}_{n,N}) &= S(\mathbf{P}, k, m, \mathbb{N}_{n,N} \setminus \{m\}) \cdot (I + S(\mathbf{P}, m, m, \mathbb{N}_{n,N}), \qquad \text{and} \\ S(\mathbf{P}, 0, n, \mathbb{N}) &= S(\mathbf{P}, 0, n, \mathbb{N} \setminus \{n\}) \cdot (I + S(\mathbf{P}, n, n, \mathbb{N})). \end{aligned}$$

In both cases, the first S-series on the right-hand side is a hitting probability, and thus convergent, the convergence of the second S-series on the right-hand side is proved similar to that of $S(\mathbf{P}, n, n, \mathbb{N}_{n,N})$. With Theorem 5.1.2, we obtain convergence of v, and $v(I, \mathbf{H})$ is the minimal solution (with equality) of

$$\sum_{k=0}^{\infty} \psi_k p_{km} \le \psi_m, \qquad m \in \mathbb{N}.$$

The interpretation is a direct consequence of $v_n(I, \mathbf{H}) = S(\mathbf{P}, 0, n, \mathbb{N})$ and the stochastic interpretation of the S-series. Alternatively, let

$$F(x, B) = \sum_{k=0}^{\infty} \mathbb{P}_x(\tau_{E_0} > k,\ X_k \in B)$$

as defined in Theorem C.2.4. Then $F(x, B) = \sum\limits_{n=0}^{\infty} F(x, B \cap E_n) =: \sum\limits_{n=0}^{\infty} F_n(x, B)$ with $F_0 = \mathrm{id}_{E_0}$ and

$$\sum_{k=0}^{\infty} F_k p_{km} = F_m, \qquad m \in \mathbb{N}.$$

Furthermore, (F_k) is the minimal solution for this system of equations, see Theorem C.2.4. Hence, $v_n(I, \mathbf{H}) = F_n$.

b) Follows directly from $v(\psi_0, \mathbf{H}) = \psi_0 v(I, \mathbf{H})$.

c) Due to $S(\mathbf{P}, 0, 0, \mathbb{N}) = T$, we have convergence of $S(\mathbf{P}, 0, 0, \mathbb{N})$, and thus, $\mathrm{gcf}\,(\mathbf{H})$ converges with $\mathrm{gcf}\,(\mathbf{H}) = \mathrm{id}_{E_0} - T$. T is $\varphi|_{E_0}$-irreducible (see Theorem C.2.4), and obviously $\psi_0 T \overset{(\le)}{=} \psi_0 \;\Leftrightarrow\; \psi_0 \mathrm{gcf}\,(\mathbf{H}) \overset{(\ge)}{=} 0$. Now assume that ψ_0 is an arbitrary σ-finite measure with $\psi_0 T \le \psi_0$. Then still $\psi_0 \mathrm{gcf}\,(\mathbf{H}) \ge 0$, and with $\psi = v(\psi_0, \mathbf{H})$, we have $\psi \mathbf{H} \ge 0$, that is $\psi \mathbf{P} \le \psi$, and hence, ψ is subinvariant (and still σ-finite). Since we assume that E_n is petite, $\psi(E_n) < \infty$, that is, $\psi_n = \psi|_{E_n}$ is a finite measure automatically. In particular, ψ_0 is finite, and since finite subinvariant measures are invariant (see Appendix C), $\psi_0 T = \psi_0$. In total, for T, every subinvariant measure is invariant, and since in case of transience there was a strictly subinvariant measure, T is recurrent. Furthermore, $\psi_0 T = \psi_0$ yields $\psi \mathbf{P} = \psi$, that is, ψ is an invariant measure for $\mathbf{P}$.

d) We have $\mathrm{gcf}\,(\mathbf{H}_f) = S\left(\tilde{\mathbf{P}}_f, 0, 0, \mathbb{N}\right)$, where

$$\tilde{\mathbf{P}}_f = \begin{pmatrix} \mathrm{id}_{E_0} - \psi_0 f_0 & -\psi_0 p_{01} & -\psi_0 p_{02} & \dots \\ -f_1 & \mathrm{id}_{E_1} - p_{11} & -p_{12} & \dots \\ -f_2 & -p_{21} & \mathrm{id}_{E_2} - p_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \ge 0,$$

and hence, the approximants $K_{f,0}^{(N)}$ increase monotonically. Thus, $\psi_0 \mathrm{gcf}\,(\mathbf{H}_f)$ either converges or diverges to ∞. So does $\int_E \psi(dx) f(x)$, and we have

$$\int_E \psi(dx) f(x) = \sum_{n=0}^{\infty} \psi_n f_n = \psi_0 f_0 + \sum_{n=1}^{\infty} \psi_0 R_{0,1,n} f_n = \mathrm{gcf}\,(\mathbf{H}_f)\,.$$

e) Additionally, put $g_0 = 1$. With $\tilde{p}_{m0} = f_{m0}$ and $\tilde{p}_{mn} = p_{mn}$ for $n \geq 1$, we obtain $\sum_{k=0}^{\infty} \tilde{p}_{mk} g_k \leq g_m$, and with Theorem 5.1.1, we obtain

$$\begin{aligned}
0 &\leq \mathrm{gcf}\,(\mathbf{H}_f) - K_{f,0}^{(N)} = \tilde{K}_{f,0}^{(N)} - \mathrm{gcf}\left(\tilde{\mathbf{H}}_f\right) \\
&\leq \sum_{m=N+1}^{\infty} S\left(\tilde{\mathbf{P}}, 0, m, \mathbb{N}_{1,N}\right) g_m \\
&= \sum_{m=N+1}^{\infty} \psi_0 p_{0m} g_m + \sum_{n=1}^{N} \sum_{m=N+1}^{\infty} \psi_0 S(\mathbf{P}, 0, n, \mathbb{N}_{1,N}) p_{nm} g_m \\
&= \sum_{m=N+1}^{\infty} \psi_0 p_{0m} g_m + \sum_{n=1}^{N} \psi_n^{(N)} \sum_{m=N+1}^{\infty} p_{nm} g_m.
\end{aligned}$$

□

5.3 Application to continuous-time Markov chains

Now, we consider continuous-time Markov chains (abbreviated by CTMC), see section C.3 for an introduction.

Let $Y = (Y_t)_{t \geq 0}$ be a CTMC with values in $(E, \mathcal{P}(E))$, where $E = E_0 \cup E_1 \cup \ldots$ with finite sets $E_1, E_2, \ldots$, $d_n = |E_n|$, and let $\mathbf{Q} = (q_{mn})_{m,n \in \mathbb{N}_0}$ be the generator matrix, $q_{mn} \in \mathbb{R}^{d_m \times d_n}$. Note that finiteness implies petiteness (for the embedded jump chain). Furthermore, we will assume $x \to E_0$ for all $x \notin E_0$, which implies that none state $x \notin E_0$ is absorbing. Since for CTMCs, absorption probabilities, mean absorbing times, and invariant measures meet the same minimality properties as for DTMPs, we can extend our results quite easily to CTMCs.

Theorem 5.3.1. *Let Y be a CTMC with state space $E = \dot{\bigcup}_{n \in \mathbb{N}_0} E_n$, where E_n is finite for $n \geq 1$. For $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, let q_{mn} be the block of transition rates between states $\in E_m$ and states $\in E_n$, and let $\alpha_n = (\mathbb{P}(Y_0 = x))_{x \in E_n}$. Let $x \to E_0$ for all $x \in E \setminus E_0$, and let*

$$\mathbf{H} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \ldots \\ -q_{10} & -q_{11} & -q_{12} & \ldots \\ -q_{20} & -q_{21} & -q_{22} & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

a) $w(I,\mathbf{H})$ converges with the interpretation

$$w_n(I,\mathbf{H})(x,y) = \mathbb{P}_x(\sigma_{E_0} < \infty, Y_{\sigma_{E_0}} = y), \qquad x \in E_n,\ y \in E_0.$$

b) $\operatorname{gcf}(\mathbf{H})$ converges as a non-negative d_0-dimensional row vector with the interpretation

$$\operatorname{gcf}(\mathbf{H})(y) = \mathbb{P}(\sigma_{E_0} < \infty, Y_{\sigma_{E_0}} = y), \qquad y \in E_0,$$

c) If there is a sequence $(\mu_n)_{n\in\mathbb{N}}$ of d_n-dimensional row vectors μ_n with

$$\alpha_m + \sum_{k=1}^{\infty} \mu_k q_{km} \le 0, \qquad m \in \mathbb{N},$$

we have

$$0 \le \operatorname{gcf}(\mathbf{H}) - K_0^{(N)} \le \sum_{m=N+1}^{\infty} \mu_m q_{m0} + \sum_{n=1}^{N} \sum_{m=N+1}^{\infty} \mu_m q_{mn} w_n^{(N)}.$$

d) Let $(F,\mathcal{F})$ be a Polish space, let $h \ge 0$ be a kernel on $E_0 \times \mathcal{F}$ with $||h|| < \infty$ and replace p_{n0} by $p_{n0}h$ and α_0 by $\alpha_0 h$. Then the convergence and speed-of-convergence statements in a), b) and c) still hold, and we have the interpretations

$$\begin{aligned} w_n(I,\mathbf{H})(x,B) &= \int \mathbb{P}_x(\sigma_{E_0} < \infty, Y_{\sigma_{E_0}} \in dy) h(y,B), \qquad x \in E_n,\ B \in \mathcal{F}, \\ \operatorname{gcf}(\mathbf{H})(B) &= \int \mathbb{P}(\sigma_{E_0} < \infty, Y_{\sigma_{E_0}} \in dy) h(y,B), \qquad B \in \mathcal{F}. \end{aligned}$$

Proof. Let $\lambda_0 = \mathrm{id}_{E_0}$ and $\lambda_n = \operatorname{diag}\left(\frac{1}{-q_{xx}}\right)_{x\in E_n}$ for $n \ge 1$. Since any $x \notin E_0$ is non-absorbing, λ_n is invertible, and applying this equivalence transformation yields the matrix $\mathbf{H}$ from Theorem 5.2.1, where p_{mn} are the block-transition probabilities for the embedded jump chain. Hence, all convergence and speed-of-convergence statements directly follow. The stochastic interpretation follows from Theorem C.3.1. Alternatively, we can use Theorem 5.2.1, since the equivalence transformation has no impact on the solution, and the hitting probabilities for the CTMC and its embedded jump chain coincide. □

Theorem 5.3.2. *Let Y be a CTMC with state space $E = \dot{\bigcup_{n\in\mathbb{N}_0}} E_n$, where E_n is finite for $n \ge 1$. For $m,n \in \mathbb{N}$, let q_{mn} be the block of transition rates between states $\in E_m$ and states $\in E_n$, and let $\alpha_n = (\mathbb{P}(X_0 = x))_{x\in E_n}$. Let $x \to E_0$ for all $x \in E \setminus E_0$, and let*

$$\mathbf{H} = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \dots \\ -f_1 & -q_{11} & -q_{12} & \dots \\ -f_2 & -q_{21} & -q_{22} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where $f_n \ge 0$ is a d_n-dimensional column vector.

a) We have

$$w_n(I,\mathbf{H})(x) = \mathbb{E}_x\left[\int_0^{\sigma_{E_0}} f(Y_s)ds\right], \qquad x \in D_n,\ n \in \mathbb{N},$$

where both sides take values in $[0,\infty]$.

b) We have

$$\operatorname{gcf}(\mathbf{H}) = \mathbb{E}\left[\int_0^{\sigma_{E_0}} f(Y_s)ds\right],$$

where both sides take values in $[0,\infty]$.

Proof. We use the same equivalence transformation as in the proof of Theorem 5.3.1. Then the convergence statement follows from Theorem 5.2.2, and the stochastic interpretation is a consequence of the minimality condition in Theorem C.3.2. □

Theorem 5.3.3. *Let Y be a CTMC with state space $E = \dot{\bigcup_{n\in\mathbb{N}_0}} E_n$, where E_n is finite for all $n \in \mathbb{N}_0$. For $m,n \in \mathbb{N}_0$, let q_{mn} be the block of transition rates between states $\in E_m$ and states $\in E_n$, let φ be an irreducibility measure with $\varphi(E_0) > 0$, let E_0 be Harris recurrent, and let*

$$\mathbf{H} = \begin{pmatrix} -q_{00} & -q_{01} & -q_{02} & \cdots \\ -q_{10} & -q_{11} & -q_{12} & \cdots \\ -q_{20} & -q_{21} & -q_{22} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \qquad \textit{and}$$

$$\mathbf{H}_f = \begin{pmatrix} \psi_0 f_0 & -\psi_0 q_{01} & -\psi_0 q_{02} & \cdots \\ f_1 & -q_{11} & -q_{12} & \cdots \\ f_2 & -q_{21} & -q_{22} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where $\psi_0 \geq 0$ is a d_0-dimensional row vector, (f_n) is a sequence of d_n-dimensional column vectors $f_n \geq 0$, $f : E \to \mathbb{R}$ is defined by $f(x) = f_n(x)$ for $x \in E_n$.

a) $v(I,\mathbf{H})$ converges with the interpretation

$$v_n(I,\mathbf{H})(x,y) = \frac{1}{-q_{xx}}\int_0^\infty \mathbb{P}_x(\tau_{E_0} > s,\ Y_s = y)ds, \qquad x \in E_0,\ y \in E_n.$$

b) $\psi = v(\psi_0,\mathbf{H}) = v\left(\psi_0,\overline{\mathbf{H}}\right)$ converges for any d_0-dimensional row vector $\psi_0 \geq 0$. Then, $\psi_n \geq 0$ is a d_n-dimensional row vector, and the sequence (ψ_n) satisfies

$$\sum_{k=0}^\infty \psi_k p_{km} = \psi_m, \qquad m \in \mathbb{N}.$$

c) $\operatorname{gcf}(\mathbf{H})$ converges and $-\operatorname{gcf}(\mathbf{H})$ is a $\varphi|_{E_0}$-irreducible and finite (and hence recurrent) conservative generator matrix. Invariance of ψ_0 for T is equivalent to $\operatorname{gcf}(\mathbf{H}) = 0$ and implies invariance of ψ for $\mathbf{Q}$.

d) For any d_0-dimensional row vector ψ_0 we can interprete

$$\operatorname{gcf}(\mathbf{H}_f) = \int_E \psi(dx) f(x),$$

where $\psi = v(\psi_0, \mathbf{H})$.

e) If there is a sequence $(g_n)_{n\in\mathbb{N}}$ of d_n-dimensional column vectors $g_n \geq 0$ with

$$f_m + \sum_{k=1}^{\infty} q_{mk} g_k \leq 0, \qquad m \in \mathbb{N},$$

we have

$$0 \leq \operatorname{gcf}(\mathbf{H}_f) - K_{f,0}^{(N)} \leq \sum_{m=N+1}^{\infty} \psi_0 q_{0m} g_m + \sum_{n=1}^{N} \psi_n^{(N)} \sum_{m=N+1}^{\infty} q_{nm} g_m,$$

where $K_{f,0}^{(N)}$ is the Nth approximant for $\operatorname{gcf}(\mathbf{H}_f)$ and $\psi_n^{(N)} = v_n^{(N)}(\psi_0, \mathbf{H})$.

Proof. In principle, define λ_n as in the proof of Theorem 5.3.1, but note that in E_0, there may be (at most) one absorbing state x. In this case, replace the diagonal entry of λ_n by 1. Then λ_n is still invertible, and the application of the equivalence transformation gives the transition probability matrix (p_{mn}) of the embedded jump chain. Theorem C.3.4 and Theorem 5.2.3 yield a)-d).

For proving e), we define $\tilde{\psi}_n = \psi_n \lambda_n^{-1}$, $\tilde{f}_n = \lambda_n f_n$ and $p_{mk} = \lambda_m q_{mk} + \delta_{mk}$. Then $f_m + \sum_{k=1}^{\infty} q_{mk} g_k \leq 0$ yields $\tilde{f}_m + \sum_{k=1}^{\infty} p_{mk} g_k \leq g_m$, and hence, we can apply Theorem 5.2.3. Note that due to the choices of $\tilde{\psi}$, $\tilde{f}$ and p_{mk}, the error estimate directly applies to $\operatorname{gcf}(\mathbf{H}_f)$ as defined here. We obtain

$$\begin{aligned} 0 \leq \operatorname{gcf}(\mathbf{H}_f) - K_{f,0}^{(N)} &\leq \sum_{m=N+1}^{\infty} \tilde{\psi}_0 p_{0m} g_m + \sum_{n=1}^{N} \tilde{\psi}_n^{(n)} \sum_{m=n+1}^{\infty} p_{nm} g_m \\ &= \sum_{m=N+1}^{\infty} \psi_0 q_{0m} g_m + \sum_{n=1}^{N} \psi_n^{(N)} \sum_{m=N+1}^{\infty} q_{nm} g_m. \end{aligned}$$

□

5.4 Literature review

We already emphasized on the fact that computing invariant distributions for DTMCs or CTMCs with finite state space is possible by performing LU decomposition without pivoting, see [Ste09] for the scalar case and [GH90] for the block-matrix case. Similarly, the system of equations for computing hitting probablities or mean hitting times can be solved by LU decomposition without pivoting since we have the same matrix in our system of equations. Hence, for finite systems, the results of this chapter are quite well-known. Nevertheless, the

direct computation of the total hitting probabilities (by using the initial distribution α in Theorems 5.2.1, 5.2.2, 5.3.1 and 5.3.2) and the direct computation of ψf (in Theorems 5.2.3 and 5.3.3) and its interpretation as gcf is new, and results in memory-efficient algorithms, see chapter 6.

The minimality of invariant measures or invariant distributions was explicitly and extensively used by Neuts [Neu81] when introducing matrix-geometric methods for solving Markov chains. Basically, Neuts considered the problem of determining the invariant measure $\pi = (\pi_n)_{n\in\mathbb{N}}$ for block-generator matrices

$$\mathbf{Q} = \begin{pmatrix} B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & \\ B_2 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

His main result was that the invariant distribution π is uniquely determined by

- $\pi_n = \pi_0 R^n$,
- $\pi_0 \sum_{n=0}^{\infty} R^n B_n = 0$,
- $\pi_0 (I - R)^{-1} (1, \ldots, 1)^T = 1$,

where R is the minimal non-negative solution of the matrix equation

$$\sum_{n=0}^{\infty} R^n A_n = 0.$$

In chapter 6, we will consider the special case of lower Hessenberg matrices. We will point out that in this case, the gcf-method yields $\pi_n = \pi_{n-1} R_{n-1,n,n}$, and due to the *level-independency* (that is, up to the first column of $\mathbf{Q}$, q_{mn} only depends on $n - m$), $R_{n-1,n,n}$ does not depend on n. Thus, the gcf-method yields $\pi_n = \pi_0 R^n$ with $R = R_{0,1,1} = v_1(I, \mathbf{H})$. Hence, it does not surprise that Neuts interpreted the entries $R(x, y)$ of R as

$$R = \frac{1}{-q_{xx}} \int_0^{\infty} \mathbb{P}_x(\tau_{E_0} > s,\ Y_s = y) ds, \qquad x \in E_{n-1},\ y \in E_n$$

for some $n \in \mathbb{N}$, which exactly corresponds to our interpretation in Theorem 5.3.3.

Later, Bright and Taylor [BT95] used the property of minimality again. They proposed an algorithm for computing the stationary distribution π for *level-dependent quasi-birth-death processes*, that is, CTMCs with block-tridiagonal generator

$$\mathbf{Q} = \begin{pmatrix} q_{00} & q_{01} & & & \\ q_{10} & q_{11} & q_{12} & & \\ & q_{21} & q_{22} & q_{23} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

For their algorithm, they used that

$$\pi_n = \pi_{n-1} R_{n-1},$$

where R_{n-1} is the minimal solution of the matrix equation

$$q_{n-1,n} + R_{n-1}q_{nn} + R_{n-1}R_n q_{n+1,n} = 0,$$

(up to notation, this is equation (1.2) in [BT95]). Again, $R_{n-1} = R_{n-1,n,n}$ with our notation, and again, Bright and Taylor gave an interpretation for the entries of R_n which is quite similar to that of $v_n(I, \mathbf{H})$ in Theorem 5.3.3.

Hanschke [Han99] used a similar algorithm for computing the invariant distribution of a level-dependent quasi-birth-death process, and explicitly stated the relationship between this algorithm and (non-generalized) matrix-valued continued fractions. Furthermore, he discussed the important topic of subdominance of the invariant disbribution: There are other solutions of $x_{n-1}q_{n-1,n} + x_n q_{nn} + x_{n+1}q_{n+1,n} = 0$ which dominate π asymptotically. We will come back to such considerations in chapter 8.

In principle, more algorithms for computing invariant distributions are based on similar considerations, we will refer to some of them at the end of chapter 6 which is dedicated to algorithms.

In the scalar case $q_{ij} \in \mathbb{R}$, the relationship between Markov Chains and gcfs was established by Hanschke [Han92], where gcfs in the sense of de Bruins n-fractions [Bru74; Bru78] were used. In the scalar case, de Bruins n-fractions are quite strongly related to our definition of gcfs, see section 7.4.2 for more details.

In total, the results of this chapter generalize results for the representation of invariant distributions for irreducible transition probability matrices or generator matrices with certain (block) structures. Furthermore, by considering the transposed system, we have extended them to the computation of hitting probabilities and mean hitting times.

Finally, we have to comment on the use of Lyapunov functions in the context of error estimates for Markov chains. Lyapunov functions have been used for finding upper bounds for ψf with some function $f \geq 0$:

- Glynn and Zeevi [GZ08] studied the case of positive recurrent Markov processes (discrete time and general state space, continuous time and discrete state space, stochastic differential equations, jump diffusion processes). For positive recurrent CTMCs, their main result was as follows: Let π be the invariant distribution, let $g \geq 0$ for some function g and let Qg be bounded. Then $\pi Qg \geq 0$. Hence, if $Qg \leq -f + c\mathbf{1}$ for some function $f \geq 0$ and some $c \in [0, \infty)$, we have $\pi f \leq c$.
- Dayar et al [Day+11] used a similar concept for finding finite sets $C \subset E$ with

  $$\sum_{i \in C} \pi_i \geq 1 - \alpha$$

  for some (small) $\alpha > 0$. Here, $Qg \leq c\mathbf{1} - (\gamma + c)\mathbf{1}_{E \setminus C}$ guarantees that

  $$(\gamma + c) \sum_{i \in E \setminus C} \pi_i \leq c,$$

  that is,

  $$\sum_{i \in C} \pi_i \geq 1 - \frac{c}{\gamma + c}.$$

- In [BS14] and [BS15], the general recurrent case was considered (for DTMCs and CTMCs). If ψ is an invariant measure, $g \geq 0$ with $Qg \leq h$ for some ψ-integrable $h \geq 0$, we still have $\psi Qg \geq 0$. In particular, if $Qg \leq -f + h$ for ψ-integrable functions $f, h \geq 0$, we have $\psi f \leq \psi h$. With a consideration similar to that in [Day+11], we obtain

$$\frac{\sum\limits_{i \in C} \psi_i f(i)}{\sum\limits_{i \in E} \psi_i f(i)} \geq 1 - \frac{c}{\gamma + c}$$

  in case that

$$\sum_{j \in E} q_{ij} g(j) \leq f(i) \left(c - (c + \gamma) \mathbf{1}_{E \setminus C}(i) \right).$$

However, all these methods directly refer to estimating the error when approximating $\psi f \approx \sum\limits_{i \in C} \psi_i f(i)$, where C is a finite set and ψ is the exact invariant measure. For theoretical purposes, this may yield useful results, but since we cannot compute ψ_i for $i \in C$ exactly, these estimates do not help directly in the context of computing ψf numerically.

To the knowledge of the author, the estimates for $\psi f - \psi^{(N)} f = \operatorname{gcf}(\mathbf{H}_f) - K_{f,0}^{(N)}$ by means of Lyapunov functions are new in the context of Markov chains (and in the more general theory developed in chapter 4). In Example 6.2.1, we will demonstrate that the bound which is obtained by the Lyapunov function may be quite good. On the other hand, it may become hard to 'guess' an appropriate function g in more complicated models.

Chapter 6

Algorithms for Markov chains

In this chapter, we will discuss the algorithmic computation of absorption probabilities and stationary characteristics of Markov chains with discrete state spaces, that is DTMCs or CTMCs with state space $E = \dot{\bigcup}_{i \in \mathbb{N}_0} E_i$, where $E_1, E_2, \ldots$ are finite. The results in section 5.2 and 5.3 were quite similar, and any algorithm for CTMCs with block generator matrix $\mathbf{Q} = (q_{mn})$ can be interpreted as an algorithm for DTMCs with block transition probability matrix $\mathbf{P} = (p_{mn})$ by setting $q_{mn} = p_{mn} - \delta_n I_{d_n}$, where $d_n = |E_n|$, and I_{d_n} is the $d_n \times d_n$-identity matrix.

The results of the previous chapter tell us that approximations for absorption probabilities, mean absorbing times, and invariant measures are given by approximants for gcfs or the gcf-based solutions v and w. The computation of these approximants, that is, $v_n^{(N)}$ and $w_n^{(N)}$ is directly related to (block-)Gaussian elimination or LU decomposition (see section 4.10). Furthermore, in the literature, there are approaches for computing invariant distributions for Markov chains with a special structure (e.g. block-tridiagonal). At the end of the chapter, we will refer to such literature.

Although we will restate a few of these algorithms here, we emphasize on important issues in which most of the methods we will suggest in the next sections differ from algorithms found in the literature:

- Usually, in a first step, an invariant measure ψ (or an approximation) is computed. Afterwards, ψf is calculated for some function f.

- Similarly, most literature suggests to (approximately) compute absorption probabilities like $\mathbb{P}(\tau_0 < \infty | X_0 = n)$ by solving the corresponding system of linear equations. Afterwards, for a given initial distribution, the total probability of ever reaching state 0 is calculated by total probability.

- In chapter 5, we have demonstrated that ψf and $\mathbb{P}(\sigma_{E_0} < \infty, X_{\sigma_{E_0}} \in \cdot)$ can be written as gcfs.

- In many of the algorithms which we will state in this chapter, we will evaluate the appropriate gcf (or an approximant $K_0^{(N)}$) directly without computing the detailed measure ψ or all the absorption probabilities.

- This direct computation of $K_0^{(N)}$ can be organized memory-efficient in many situations.
- Furthermore, our results in chapter 5 include error bounds. Hence, we are able to find computable lower and upper bounds for ψf (or the total hitting probability).
- It will turn out that the simultaneous computation of both lower and upper bound requires little more effort than the computation of $K_0^{(N)}$ (which is the lower bound).

In the next sections, we will derive algorithms for some special structures of $\mathbf{Q}$ (or $\mathbf{P}$) which often occur in practical applications. Naturally, many more structures could be considered.

6.1 Absorption and hitting probabilities

For CTMCs, we consider the recursions

$$K_n^{(N)} = -q_{nn} - \sum_{m=n+1}^{N} q_{nm} L_{m,n+1,n}^{(N)}, \tag{6.1.1}$$

$$L_{n,n,k}^{(N)} = \left(K_n^{(N)}\right)^{-1} \left(q_{nk} + \sum_{m=n+1}^{N} q_{nm} L_{m,n+1,k}^{(N)}\right), \quad k = 1, \ldots, n-1, \tag{6.1.2}$$

$$L_{n,n,0}^{(N)} = \left(K_n^{(N)}\right)^{-1} \left(q_{n0} h + \sum_{m=n+1}^{N} q_{nm} L_{m,n+1,0}^{(N)}\right), \tag{6.1.3}$$

$$L_{m,n,k}^{(N)} = L_{m,n+1,k}^{(N)} + L_{m,n+1,n}^{(N)} L_{n,n,k}^{(N)}, \quad k = 0, \ldots, n-1, \quad m = n+1, \ldots, N, \tag{6.1.4}$$

which are iterated from $n = N$ down to $n = 1$, and finally, we have

$$K_0^{(N)} = \alpha_0 h + \sum_{m=1}^{N} \alpha_m L_{m,1,0}^{(N)}.$$

Then according to Theorem 5.3.1, the matrix $L_{n,1,0}^{(N)}$ is an approximation for

$$\beta_n := \left(\sum_{z \in E_0} \mathbb{P}_x(\tau_{E_0} < \infty, X_{\tau_{E_0}} = z) h(z, y) \right)_{x,y},$$

and $K_0^{(N)}$ is an approximation for

$$B := \left(\sum_{z \in E_0} \mathbb{P}(\tau_{E_0} < \infty, X_{\tau_{E_0}} = z) h(z, y) \right)_{y}.$$

For finite E_0, it makes sense to choose $h = I_{d_0}$.

6.1.1 Algorithms for computing conditional hitting probabilities

We start with considering block-tridiagonal generators $\mathbf{Q}$, that is $q_{mn} = 0$ for $|m-n| \geq 2$. Then, the recursion for L simplifies to

$$\begin{aligned} L^{(N)}_{n,n,n-1} &= \left(-q_{nn} - q_{n,n+1}L^{(N)}_{n+1,n+1,n}\right)^{-1} q_{n,n-1}, \qquad n \geq 2 \\ \beta^{(N)}_1 &= L^{(N)}_{1,1,0} = \left(-q_{11} - q_{12}L^{(N)}_{2,2,1}\right)^{-1} q_{10}h, \\ \beta^{(N)}_n &= L^{(N)}_{n,n,n-1}\beta^{(N)}_{n-1}, \qquad n \geq 2. \end{aligned}$$

Algorithm 6.1.1. Let $\mathbf{Q}$ be block-tridiagonal, and choose N large.

- Set $L_N = 0$.
- For $n = N, N-1, \ldots, 2$, compute $L_{n-1} = \left(-q_{nn} - q_{n,n+1}L_n\right)^{-1} q_{n,n-1}$.
- Compute $\beta_1 = \left(-q_{11} - q_{12}L_1\right)^{-1} q_{10}h$.
- For $n = 2, \ldots, N$ compute $\beta_n = L_{n-1}\beta_{n-1}$.

Next, we consider upper block-Hessenberg matrices, that is $q_{mn} = 0$ for $n \leq m-2$ (block-$M/G/1$-type matrices). Here, we have the recursion

$$L^{(N)}_{n,n,n-1} = \left(-q_{nn} - \sum_{m=n+1}^{N} q_{nm} \prod_{n+1}^{r=m} L^{(N)}_{r,r,r-1}\right)^{-1} q_{n,n-1} \qquad \text{and} \qquad \beta^{(N)}_n = L^{(N)}_{n,n,n-1}\beta^{(N)}_{n-1}$$

and we suggest the following algorithm.

Algorithm 6.1.2. Let $\mathbf{Q}$ be upper block-Hessenberg, and let N be chosen large.

- Set $L_N = 0$ and initialize $K_k = -q_{kN}$ for $k = 1, \ldots, N$.
- For $n = N, N-1, \ldots, 2$, compute $L_{n-1} = K_n^{-1} q_{n,n-1}$ and update $K_k = -q_{k,n-1} + K_k L_{n-1}$ for $k = 1, \ldots, n-1$.
- Compute $\beta_1 = K_1^{-1} q_{10}h$.
- For $n = 2, \ldots, N$ compute $\beta_n = L_{n-1}\beta_{n-1}$.

Finally, we consider lower block-Hessenberg matrices, that is $q_{mn} = 0$ for $n \geq m+2$. Here, we have

$$\begin{aligned} L^{(N)}_{n,n,k} &= \left(-q_{nn} - q_{n,n+1}L^{(N)}_{n+1,n+1,n}\right)^{-1} \left(q_{nk} + q_{n,n+1}L^{(N)}_{n+1,n+1,k}\right) \qquad \text{and} \\ \beta^{(N)}_n &= \sum_{k=0}^{n-1} L^{(N)}_{n,n,k}\beta^{(N)}_k, \end{aligned}$$

and we suggest

Algorithm 6.1.3. Let $\mathbf{Q}$ be lower block-Hessenberg, and let N be chosen large.

- Set $L_{N+1,k} = 0$ for $k = 0, \ldots, N$.
- For $n = N, N-1, \ldots, 1$, compute

$$\begin{aligned} L_{n,k} &= (-q_{nn} - q_{n,n+1}L_{n+1,n+1,n})^{-1} (q_{nk} + q_{n,n+1}L_{n+1,n+1,k}), \qquad k = 1, \ldots, n-1, \\ L_{n,0} &= (-q_{nn} - q_{n,n+1}L_{n+1,n+1,n})^{-1} (q_{n0}h + q_{n,n+1}L_{n+1,n+1,0}). \end{aligned}$$

- Set $\beta_1 = L_{1,0}$, and for $n = 2, \ldots, N$, compute $\beta_n = L_{n,0} + \sum_{k=1}^{n-1} L_{n,k}\beta_k$.

In case $d_n = d$ (for $n \geq 1$), we briefly consider the complexity of the algorithms: For block-tridiagonal generators, the computational effort is $O(Nd^3)$, and storage requirement is $O(Nd^2)$. For upper Hessenberg matrices, the computational effort is $O(N^2d^3)$, the storage requirement is $O(Nd^2)$. For lower Hessenberg matrices, the computational effort ist $O(N^2d^3)$, the storage requirement is $O(N^2d^2)$. Since in principle, the method is equivalent to Gaussian elimination (without pivoting), these considerations do not surprise.

6.1.2 Algorithms for computing total hitting probabilities

Now we consider the approximation of B by $K_0^{(N)}$. Here, we may save storage amount by applying appropriate Horner-type schemes. For example, in case of a tridiagonal $\mathbf{Q}$, we have

$$K_0^{(N)} = \alpha_0 h + \left(\alpha_1 + \left(\alpha_2 + \left(\ldots \left(\alpha_{N-1} + \alpha_N L^{(N)}_{N,N,N-1}\right) L^{(N)}_{N-1,N-1,N-2}\right) \ldots\right) L^{(N)}_{2,2,1}\right) L^{(N)}_{1,1,0},$$

Algorithm 6.1.4. Let $\mathbf{Q}$ be tridiagonal, and let N be chosen large.

- Set $B = \alpha_N$ and $L = 0$.
- For $n = N, N-1, \ldots, 2$ replace $L = (-q_{nn} - q_{n,n+1}L)^{-1} q_{n,n-1}$ and $B = \alpha_{n-1} + BL$.
- Replace $B = \alpha_0 h + B(-q_{11} - q_{12}L)^{-1} q_{10}h$, and return B.

Similar considerations yield

Algorithm 6.1.5. Let $\mathbf{Q}$ be upper block-Hessenberg, and let N be chosen large.

- Initialize $B = \alpha_N$ and $K_k = -q_{kN}$ for $k = 1, \ldots, N-1$.
- For $n = N, N-1, \ldots, 2$
  - $L = K_n^{-1} q_{n,n-1}$.
  - Replace $B = \alpha_{n-1} + BL$.
  - Replace $K_k = -q_{k,n-1}K_k L$ for $k = 1, \ldots, n-1$.

- Compute $L = K_1^{-1} q_{10} h$, replace $B = \alpha_0 h + BL$ and return B.

Algorithm 6.1.6. Let $\mathbf{Q}$ be lower block-Hessenberg, and let N be chosen large.

- Initialize $B_k = \alpha_k$ for $k = 1, \ldots, N$, $B_0 = \alpha_0 h$ and $L_k = 0$ for $k = 0, \ldots, N$.
- For $n = N, N-1, \ldots, 1$
  - Compute the inverse of $K = -q_{nn} - q_{n,n+1} L_n$.
  - Replace $L_k = K^{-1}(q_{nk} + q_{n,n+1} L_k)$ for $k = 1, \ldots, n-1$ and $L_0 = K^{-1}(q_{n0} h + q_{n,n+1} L_0)$.
  - Replace $B_k = B_k + B_n L_k$ for $k = 0, \ldots, n-1$.
- Return $B = B_0$.

For a brief anaylsis of the algorithms, let $d_n = d$ be constant again. For upper Hessenberg matrices, the storage requirement is still $O(N)$ (the coefficient is smaller) so there is no great advantage of the new algorithm. For lower Hessenberg matrices, the storage requirement is $O(N)$ instead of $O(N^2)$, and for tridiagonal matrices, the storage requirement is $O(1)$ instead of $O(N)$. Hence, in these situations, the direct algorithms for computing the total hitting probabilities are much more efficient.

Remark 6.1.1. Note that the effort only slightly increases when computing B for several initial distributions at the same time. In this case, we replace the row vectors α_n by matrices.

6.1.3 Error analysis

According to Theorem 5.3.1, $K_0^{(N)}$ monotonically increases to B, hence it is a lower bound for B. If we find a sequence $(\mu_n)_{n \in \mathbb{N}}$ with

$$\alpha_m + \sum_{k=1}^{\infty} \mu_k q_{km} \leq 0, \qquad m \in \mathbb{N},$$

an upper bound is given by

$$K_0^{(N)} + \sum_{m=N+1}^{\infty} \mu_m q_{m0} h + \sum_{n=1}^{N} \sum_{m=N+1}^{\infty} \mu_m q_{mn} L_{n,1,0}^{(N)}.$$

If it is possible to find exact representations for $\sum_{m=N+1}^{\infty} \mu_m q_{mn}$, we obtain an upper bound for B by replacing α_n by $\alpha_n + \sum_{m=N+1}^{\infty} \mu_m q_{mn}$ in our algorithms. Here, we just consider the situation of block-tridiagonal $\mathbf{Q}$. Then, $\alpha_n + \sum_{m=N+1}^{\infty} \mu_m q_{mn}$ simplifies to $\alpha_N + \mu_{N+1} q_{N+1,N}$ for $n = N$, and for $n < N$, it simplifies to α_n.

Algorithm 6.1.7. Let $\mathbf{Q}$ be tridiagonal, choose N large, and let $\alpha_m + \sum_{k=1}^{\infty} \mu_k q_{km} \le 0$ for all $m \in \mathbb{N}$

- Set $B = \alpha_N$, $\overline{B} = \alpha_N + \mu_{N+1} q_{N+1,N}$ and $L = 0$.
- For $n = N, N-1, \ldots, 2$ replace $L = (-q_{nn} - q_{n,n+1}L)^{-1} q_{n,n-1}$, $B = \alpha_{n-1} + BL$ and $\overline{B} = \alpha_{n-1} + \overline{B}L$.
- Replace

$$\begin{aligned} B &= \alpha_0 h + B\,(-q_{11} - q_{12}L)^{-1} q_{10} h \qquad \text{and} \\ \overline{B} &= \overline{\alpha}_0 h + \overline{B}\,(-q_{11} - q_{12}L)^{-1} q_{10} h. \end{aligned}$$

Similarly, we can use Theorem 5.3.2 for developing efficient algorithms for computing

$$\mathbb{E}\left[\int_0^{\sigma_{E_0}} f(X_s)\,ds\right].$$

In principle, we just have to replace $q_{m0}h$ by f_m, and hence, there is no need for restating these algorithms.

6.2 Computing invariant measures and long-run averages

Now, we turn to computing invariant measures ψ or long-run averages ψf. We still assume that $E = \dot{\bigcup}_{i \in \mathbb{N}_0} E_i$, and additionally, we assume that

- E_n is finite for all $n \in \mathbb{N}_0$,
- there is some irreducibility measure φ with $\varphi(E_0) > 0$ and
- E_0 is Harris recurrent.

According to Theorem 5.3.3, we have $\psi f \approx \psi_0 K_{0,f}^{(N)}$, where

$$K_n^{(N)} = -q_{nn} - \sum_{m=n+1}^{N} R_{n,n+1,m}^{(N)} q_{mn}, \tag{6.2.1}$$

$$R_{k,n,n}^{(N)} = \left(q_{kn} + \sum_{m=n+1}^{N} R_{k,n+1,m}^{(N)} q_{mn}\right)\left(K_n^{(N)}\right)^{-1}, \quad k = 0, \ldots, n-1, \tag{6.2.2}$$

$$R_{k,n,m}^{(N)} = R_{k,n+1,m}^{(N)} + R_{k,n,n}^{(N)} R_{n,n+1,m}^{(N)}, \quad k = 0, \ldots, n-1, \quad m = n+1, \ldots, N, \tag{6.2.3}$$

for $n = N, N-1, \ldots, 1$,

$$K_0^{(N)} = -q_{00} - \sum_{m=1}^{N} R_{0,1,m}^{(N)} q_{m0}, \tag{6.2.4}$$

$$\overline{K}_{0,f}^{(N)} = f_0 + \sum_{m=1}^{N} R_{0,1,m}^{(N)} f_m. \tag{6.2.5}$$

and ψ_0 being the unique (up to a constant multiple) solution of $\psi_0 \lim_{N\to\infty} K_0^{(N)} = 0$. Approximants for an invariant measure are given by $\psi_0 R_{0,1,n}^{(N)}$.

6.2.1 The computation of ψ_0

According to Theorem 5.3.3, $\lim_{N\to\infty} K_0^{(N)}$ is an $\varphi|_{E_0}$-irreducible and finite conservative generator matrix. Hence, there is a non-trivial solution ψ_0 of

$$\psi_0 \lim_{N\to\infty} K_0^{(N)} = 0,$$

ψ_0 can be chosen non-negative, and is unique up to a constant multiple.

When replacing the limit by an approximation $K_0^{(N)}$, in general, there is no non-trivial solution anymore. Hence, an important question concerns the algorithmic computation of ψ_0. There are various possibilities:

- As $N \to \infty$, $K_0^{(N)}$ converges to $\operatorname{gcf}(\mathbf{H})$ with eigenvalue 0. Hence, for sufficiently large N, there is an eigenvalue near 0. Precisely speaking, $-K_0^{(N)}$ is a non-conservative generator, and we may consider the corresponding embedded jump chain with transition probability matrix $T^{(N)}$, which is substochastic. Due to the φ-irreducibility-assumption ($-K_0^{(N)}$ and $T^{(N)}$ are $\varphi|_{E_0}$-irreducible for sufficiently large N), the absolute value of all eigenvalues is smaller than 1, and there is one eigenvalue (the Perron-Frobenius eigenvalue) which is real and positive with non-negative eigenvector, and this eigenvalue converges to 1. For $-K_0^{(N)}$, this eigenvalue is still real and positive and converges to 0 as $N \to \infty$. The corresponding eigenvector remains non-negative, and hence, it makes sense to choose this eigenvector as an approximation for ψ_0. Since the approximation depends on N, we shall write $\psi_0^{(N)}$. For details on Perron-Frobenius eigenvalues and eigenvectors, we refer to [Sen81].

- Let $x_0 \in E_0$ with $\varphi(\{x_0\}) > 0$. Since $-K_0^{(N)}$ is a finite (non-conservative) φ_{E_0}-irreducible generator, we can construct the minimal subinvariant measure $\psi_0^{(N)}$ for $-K_0^{(N)}$. This measure satisfies the invariance equations for $-K_0^{(N)}$ up to the equation which corresponds to state x_0, and in the equation which corresponds to state x_0, we have strict inequality. Fix $\psi_0^{(N)}(x_0) = 1$, then we have convergence of $\psi_0^{(N)}$ to the invariant measure ψ_0 for $-\operatorname{gcf}(\mathbf{H})$ with $\psi_0 = 1$. Furthermore, this choice of $\psi_0^{(N)}$ guarantees that the equations of $\psi^{(N)}\mathbf{Q}_{0,N} = 0$ are met up to the equation which corresponds to state x_0. Hence, $\psi^{(N)} = v^{(N)}\left(\psi_0^{(N)}, \mathbf{H}\right)$ has the interpretation as the minimal subinvariant measure for $\mathbf{Q}_{0,N}$ with $\psi^{(N)}(x_0) = 1$. Thus, $\psi^{(N)}$ converges monotonically increasing to the invariant measure ψ with $\psi(x_0) = 1$.

- By redefining the partition $E = \bigcup E_i$, we can guarantee $E_0 = \{x_0\}$, that is $|E_0| = d_0 = 1$. In this case, we know that $K_0^{(N)}$ will converge to $0 \in \mathbb{R}$, and we directly choose $\psi_0 = 1$. Obviously, $\psi^{(N)}$ has the same interpretation as in the method where we fix $\psi_0^{(N)}(x_0) = 1$.

The second and the third method will deliver the same results, and due to

- the stochastic interpretation as minimal subinvariant measures for the truncated generator $\mathbf{Q}_{0,N}$ and
- the monotone convergence of $\psi^{(N)}$ to ψ,

they have advantages in comparison to the first method. In some practical situations, the redefinition of the partition will change the structure (e.g. upper Hessenberg), and hence, we recommend the second method for determining $\psi_0^{(N)}$. Up to notations, for block-tridiagonal generator matrices $\mathbf{Q}$, this method corresponds to that in [Han99].

6.2.2 Algorithms for computing the invariant distribution

In principle, we can use the same considerations as for computing (conditional) hitting probabilities, we only have to 'transpose the algorithms'. Hence, it shall not surprise that the role of upper Hessenberg matrices and lower Hessenberg matrices interchange.

For block-tridiagonal $\mathbf{Q}$, the recursions simplify to

$$R_{n-1,n,n}^{(N)} = q_{n-1,n}\left(I - q_{nn} - R_{n,n+1,n+1}^{(N)} q_{n+1,n}\right)^{-1} \qquad \text{and} \qquad \psi_n^{(N)} = \psi_{n-1}^{(N)} R_{n-1,n,n}^{(N)}$$

and thus, we obtain

Algorithm 6.2.1. Let $\mathbf{Q}$ be block-tridiagonal, and choose N large.

- Set $R_N = 0$.
- For $n = N, N-1, \ldots, 1$ compute $R_{n-1} = q_{n-1,n}(-q_{nn} - R_n q_{n+1,n})^{-1}$.
- Set $\psi_0(x_0) = 1$ and solve $\psi_0(-q_{00} - R_0 q_{10}) = (\cdot, 0, \ldots, 0)$.
- For $n = 1, 2, \ldots, N$ compute $\psi_n = \psi_{n-1} R_{n-1}$.
- In case of positive recurrence, compute (an approximation for) the invariant distribution π by renormalizing ψ.

In slightly different forms, this algorithm was suggested in [BT95], [Han99], [BS10] [PD+10]:

- Bright and Taylor [BT95] did not use the recursion for R_n directly, and for R_N they used a non-null initialization.
- Hanschke [Han99] suggested a similar algorithm, but he represented the matrices in terms of matrix continued fractions. For approximating R_N, he used the representation of continued fractions as ordinary fractions (with some numerator and some denominator, see section 7.1).
- In [BS10], finite block-tridiagonal generators were considered, and for the first time, the recursion for R_n was used directly.

- Phung-Duc et al [PD+10] used a similar algorithm for infinite generators $\mathbf{Q}$. They demonstrated that the direct use of the recursion for the R_n outperforms the method suggested in [BT95]. For approximating R_N, they started the recursion with $R_{N_1} = 0$ for some $N_1 > N$ without storing $R_{N_1-1}, \ldots, R_{N+1}$. In principle, this initialization is equivalent to those in [BT95] and [Han99].

Naturally, the results of Algorithm 6.2.1 become better when starting with $R_{N_1} = 0$, and using the recursion $R_{n-1} = \ldots$ for computing a (better) approximation for R_N. We did not recommend this method for the following reasons:

- With enough memory, we can simply start our complete algorithm at level N_1 instead of level N. Then, we compute approximations for $\psi_{N+1}, \ldots, \psi_{N_1}$, too. In case of positive recurrence, we obtain a better approximation for the renormalization factor. If we only trust the results for $\pi_0, \ldots, \pi_N$, we obtain better results.
- If this method fails due to lack of memory, we suggest a combination of Algorithm 6.2.1 with algorithms for computing ψf directly, see Remark 6.2.1.

Next, we consider lower block-Hessenberg matrices, that is $q_{mn} = 0$ for $n \geq m + 2$ (in the context of queueing theory, block-$M/G/1$-type matrices). Here, we have $R^{(N)}_{k,n,m} = 0$ for $k \leq n - 2$ again, and we obtain $\psi^{(N)}_n = \psi^{(N)}_{n-1} R^{(N)}_{n-1,n,n}$ and

$$R^{(N)}_{n-1,n,n} = q_{n-1,n} \left(-q_{nn} - \sum_{m=n+1}^{N} \prod_{r=n+1}^{m} R^{(N)}_{r-1,r,r} q_{mn} \right)^{-1}.$$

By iteratively updating the information concerning the denominator, we obtain

Algorithm 6.2.2. Let $\mathbf{Q}$ be lower block-Hessenberg, and let N be chosen large.

- Set $R_N = 0$, and initialize $K_k = -q_{Nk}$ for $k = 0, \ldots, N$.
- For $n = N, N-1, \ldots, 1$ compute $R_{n-1} = q_{n-1,n} K_n^{-1}$ (or solve the system $R_{n-1} K_n = q_{n-1,n}$), and update $K_k = -q_{n-1,k} + R_{n-1} K_k$ for $k = 0, \ldots, n-1$.
- Set $\psi_0(x_0) = 1$ and solve $\psi_0(-q_{00} - R_0 q_{10}) = (\cdot, 0, \ldots, 0)$.
- For $n = 1, 2, \ldots, N$ compute $\psi_n = \psi_{n-1} R_{n-1}$.
- In case of positive recurrence, compute (an approximation for) the invariant distribution π by renormalizing ψ.

Finally, we consider upper block-Hessenberg matrices, that is $q_{mn} = 0$ for $n \leq m - 2$ (in the context of queueing theory, block-$GI/M/1$-type matrices). Here, the recursions simplify to

$$\psi^{(N)}_n = \sum_{k=0}^{n-1} \psi^{(N)}_k R^{(N)}_{k,n,n},$$

where

$$R^{(N)}_{k,n,n} = \left(q_{kn} + R^{(N)}_{k,n+1,n+1} q_{n+1,n} \right) \left(-q_{nn} - R^{(N)}_{n,n+1,n+1} q_{n+1,n} \right)^{-1}.$$

Algorithm 6.2.3. Let $\mathbf{Q}$ be upper block-Hessenberg, and let N be chosen large.

- Set $R_{k,N+1} = 0$ for $k = 0, \ldots, N$.
- For $n = N, N-1, \ldots, 1$, compute

$$R_{k,n}^{(N)} = \left(q_{kn} + R_{k,n+1,n+1}^{(N)} q_{n+1,n}\right)\left(-q_{nn} - R_{n,n+1}^{(N)} q_{n+1,n}\right)^{-1}, \qquad k = 0, \ldots, n-1.$$

- Set $\psi_0(x_0) = 1$ and solve $\psi_0(-q_{00} - R_{01}q_{10}) = (\cdot, 0, \ldots, 0)$.
- For $n = 1, 2, \ldots, N$, compute $\psi_n = \sum_{k=0}^{n-1} \psi_k R_{k,n,n}$.
- In case of positive recurrence, compute (an approximation for) the invariant distribution π by renormalizing ψ.

Again, the effort for the algorithms is that of block-Gaussian elimination. In case of constant level (block) sizes $d_n = d$, we can state that

- Algorithm 6.2.1 for block-tridiagonal $\mathbf{Q}$ has computational effort $O(Nd^3)$ and storage requirement $O(Nd^2)$.
- Algorithm 6.2.2 for lower block-Hessenberg $\mathbf{Q}$ has computational effort $O(N^2d^3)$ and storage requirement $O(N^2d^2)$.
- Algorithm 6.2.3 for upper block-Hessenberg $\mathbf{Q}$ has computational effort $O(N^2d^3)$ and storage requirement $O(N^2d^2)$.

6.2.3 Direct computation of long-run averages

Now we turn to directly computing an approximation for $F := \psi f$, which is important for applying the large law of numbers for Markov Chains in the form

$$\lim_{t\to\infty} \frac{\int_0^t f(X_s)ds}{\int_0^t g(X_s)ds} = \frac{\psi f}{\psi g} \qquad (\psi\,|f| < \infty,\ \psi\,|g| < \infty,\ \psi g \neq 0),$$

see Theorem C.3.5. In nearly all applications of Markov Chains, an approximation for ψ is computed, and afterwards, the multiplication ψf is performed. In what follows, for the three special cases of the structure of $\mathbf{Q}$ (block-tridiagonal, lower and upper block-Hessenberg), we demonstrate that directly computing ψf yields memory-efficient algorithms.

Again, we start with block-tridiagonal $\mathbf{Q}$. Here, we write

$$\begin{aligned}\psi f &\approx \sum_{n=0}^{N} \psi_n^{(N)} f_n = K_{0,f}^{(N)} \\ &= \psi_0^{(N)}\left(f_0 + R_{0,1,1}^{(N)}\left(f_1 + R_{1,2,2}^{(N)}\left(\ldots + R_{N-2,N-1,N-1}^{(N)}\left(f_{N-1} + R_{N-1,N,N}^{(N)} f_N\right)\right)\right)\right)\end{aligned}$$

and use a Horner-type scheme. Since we may be interested in ψf for more than one function f, we interprete $f : E \to \mathbb{R}^M$.

Algorithm 6.2.4. Let $\mathbf{Q}$ be block-tridiagonal, let $f : E \to \mathbb{R}^M$.

- Initialize $R = 0$ and $F = f_N$.
- For $n = N, N-1, \ldots, 1$ replace

$$R = q_{n-1,n}(-q_{nn} - Rq_{n+1,n})^{-1} \qquad \text{and} \qquad F = f_{n-1} + RF.$$

- Set $\psi_0(x_0) = 1$, solve $\psi_0(q_{00} + R_0 q_{10}) = (\cdot, 0, \ldots, 0)$ and compute $\psi_0 F$.

In particular, if we are interested in the long-run average

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(X_s)ds = \pi g$$

for an ergodic Markov Chain with invariant distribution π, we put $f = (g\ \mathbf{1})$, and after applying Algorithm 6.2.4, we compute $\pi g = \frac{\psi g}{\psi \mathbf{1}}$.

Remark 6.2.1. In the discussion of Algorithm 6.2.1, we mentioned that a combination of Algorithm 6.2.1 and 6.2.4 may be useful for computing good approximations of the invariant distribution for a positive recurrent Markov chain. The idea is as follows: Choose $N_1 > N$, perform Algorithm 6.2.4 with $f = \mathbf{1}$, and use the result as renormalization factor. From N down to 0, use Algorithm 6.2.1:

- Initialize $R = 0$ and $F = \mathbf{1}_{d_{N_1}}$.
- For $n = N_1, N_1 - 1, \ldots, N + 1$ replace

$$R = q_{n-1,n}(-q_{nn} - Rq_{n+1,n})^{-1} \qquad \text{and} \qquad F = \mathbf{1}_{d_n} + RF.$$

- For $n = N, N-1, \ldots, 1$ compute $R_{n-1} = q_{n-1,n}(-q_{nn} - R_n q_{n+1,n})^{-1}$ and update $F = \mathbf{1}_{d_n} + R_{n-1}F$.
- Set $\psi_0(x_0) = 1$ and solve $\psi_0(-q_{00} - R_0 q_{10}) = (\cdot, 0, \ldots, 0)$. Compute $c = \psi_0 F$ and $\pi_0 = \frac{1}{c}\psi_0$.
- For $n = 1, 2, \ldots, N$ compute $\pi_n = \pi_{n-1} R_{n-1}$.

In comparison to Algorithm 6.2.1, the storage amount does not increase, but we use the levels $N + 1, \ldots, N_1$ for obtaining a better approximation for R_N and for obtaining a better approximation of the renormalization factor.

In the algorithms suggested in [BT95], [Han99] and [PD+10], the information of the levels $N + 1, \ldots, N_1$ is only used for improving the approximation for R_N, but has no impact on the renormalization factor. Hence, our method will yield better results.

Next, we consider lower block-Hessenberg matrices $\mathbf{Q}$. Again, we have $\psi_n = \psi_{n-1} R_{n-1,n,n}$, and thus, we can use the same idea as for block-tridiagonal $\mathbf{Q}$.

Algorithm 6.2.5. Let $\mathbf{Q}$ be lower block-Hessenberg, let $f : E \to \mathbb{R}^M$.

- Initialize $F = f_N$ and $K_k = -q_{Nk}$ for $k = 0, \ldots, N$.
- For $n = N, N-1, \ldots, 1$
  - Compute $R = q_{n-1,n} K_n^{-1}$.
  - Replace $F = f_{n-1} + RF$.
  - Replace $K_k = -q_{n-1,k} + RK_k$ for $k = 0, \ldots, n-1$.
- Set $\psi_0(x_0) = 1$, solve $\psi_0 K_0 = (\cdot, 0, \ldots, 0)$ and compute $\psi_0 F$.

For upper block-Hessenberg matrices $\mathbf{Q}$, due to

$$\psi_n^{(N)} = \sum_{k=0}^{n-1} \psi_k^{(N)} R_{k,n,n}^{(N)},$$

the Horner scheme gets a bit more complicated, but it still works.

Algorithm 6.2.6. Let $\mathbf{Q}$ be upper block-Hessenberg, let $f : E \to \mathbb{R}^M$.

- Initialize $F_k = f_k$ for $k = 1, \ldots, N$, $R_k = 0$ for $k = 0, \ldots, N$ and $K = -q_{NN}$.
- For $n = N, N-1, \ldots, 1$
  - Replace $R_k = (q_{kn} + R_k q_{n+1,n}) K^{-1}$ for $k = 0, \ldots, n-1$.
  - Replace $F_k = F_k + R_k F_n$ for $k = 0, \ldots, n-1$.
  - Replace $K = -q_{n-1,n-1} - R_{n-1} q_{n,n-1}$.
- Set $\psi_0(x_0) = 1$, solve $\psi_0 K = (\cdot, 0, \ldots, 0)$ and compute $\psi_0 F_0$.

In comparison to the algorithms for computing the invariant distributions, the runtime will not change essentially, but for Algorithm 6.2.4 for block-tridiagonal $\mathbf{Q}$, the storage requirement reduces to $O(d^2)$ since there is no need for storing all R_n anymore. Similarly, the storage requirement for Algorithms 6.2.5 and 6.2.6 is $O(Nd^2)$.

6.2.4 Error analysis

Our method of choosing $\psi_0^{(N)}$ guarantees that for $f \geq 0$, $\psi^{(N)} f$ increases monotonically and converges to ψf, where ψ is the invariant measure ψ with $\psi(x_0) = 1$. Obviously, it would be of some practical interest to have an upper bound for ψf, too. Then it is quite easy to obtain computable lower and upper bounds for $\frac{\psi f}{\psi g}$ too, or in particular, for πf in the case of positive recurrence.

As for total hitting probabilities, we use the error estimate from section 5.3. Additionally, here, we have to assume that $d_0 = |E_0| = 1$ and $\psi_0 = 1$. Then Theorem 5.3.3 states

$$\operatorname{gcf}(\mathbf{H}) \leq K_{0,f}^{(N)} + \sum_{m=N+1}^{\infty} q_{0m} g_m + \sum_{n=1}^{N} R_{0,1,n}^{(N)} \sum_{m=N+1}^{\infty} q_{nm} g_m = K_{0,\overline{f}}^{(N)}$$

with $\overline{f}_n = f_n + \sum_{m=N+1}^{\infty} q_{nm} g_m$ whenever $f_m + \sum_{k=1}^{\infty} q_{mk} g_k \leq 0$. If f is m-dimensional, we have to choose g m-dimensional, too. Computing $K_f^{(N)}$ and $K_{\overline{f}}^{(N)}$ at the same time does not yield much more effort, here we just state the algorithm for block-tridiagional $\mathbf{Q}$ (where it is very easy to guarantee $d_0 = 1$).

Algorithm 6.2.7. Let $\mathbf{Q}$ be block-tridiagonal with $d_0 = |E_0| = 1$, let $f : E \to \mathbb{R}^M_{\geq 0}$, let $f_m + \sum_{k=1}^{\infty} q_{mk} g_k \leq 0$ for all $m \in \mathbb{N}$ for some $g : E \to \mathbb{R}^M_{\geq 0}$.

- Initialize $R = 0$, $F = f_N$, $\overline{F} = f_N + q_{N,N+1} g_{N+1}$.
- For $n = N, N-1, \ldots, 1$ replace

$$R = q_{n-1,n}(-q_{nn} - R q_{n+1,n})^{-1}, \qquad F = f_{n-1} + RF \qquad \text{and} \qquad \overline{F} = f_{n-1} + R\overline{F}.$$

The reason for not applying this method to partitions with $d_0 > 1$ is that we would need lower and upper approximations for ψ_0, but we only have a lower one. When applying Algorithm 6.2.7, the main task is 'guessing' a good function g.

Example 6.2.1. We consider the $M/M/1$-queue, that is

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \mu & -(\lambda+\mu) & \lambda & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

for $\rho := \frac{\lambda}{\mu} < 1$. It is well-known and easy to check that the invariant distribution is given by $\pi_n = (1-\rho)\rho^n$.

Now we demonstrate how to compute upper and lower bounds for π_0 algorithmically. First, we compute a vector $\psi^{(N)}$ with $\psi_0^{(N)} = 1$ which solves

$$\psi^{(N)} \mathbf{Q}_{0,N} = \psi^{(N)} \begin{pmatrix} -\lambda & \lambda & & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & & \\ & \mu & -(\lambda+\mu) & \lambda & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mu & -(\lambda+\mu) & \lambda & \\ & & & & \mu & -(\lambda+\mu) & \end{pmatrix}$$

up to the first equation. The unique solution is given by

$$\psi_n^{(N)} = \frac{\rho^n - \rho^{N+1}}{1 - \rho^{N+1}}.$$

For $N \to \infty$, $\psi^{(N)}$ converges monotonically increasing to the invariant distribution ψ with $\psi_0 = 1$. Hence,

$$\pi_0 = \frac{1}{\sum_{n=0}^{\infty} \psi_n},$$

and a lower bound for the denominator is given by

$$\sum_{n=0}^{N} \psi_n^{(N)} \cdot 1 = \frac{1}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}.$$

In order to obtain an upper bound, we put $g_n = \frac{n}{\mu-\lambda}$. For $m \geq 1$, we have

$$1 + \sum_{k=1}^{\infty} q_{mk} g_k \leq 1 + \frac{1}{\mu-\lambda}(\lambda(n+1) - (\lambda+\mu)n + \mu(n-1)) = 0,$$

and hence, an upper bound for $\sum \psi_n$ is given by

$$\begin{aligned} \sum_{n=0}^{N} \psi_n^{(N)} \overline{f}_n &= \sum_{n=0}^{N} \psi_n^{(N)} + \psi_N^{(N)} \lambda g_{N+1} \\ &= \frac{1}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} + \frac{\rho^N(1-\rho)}{1-\rho^{N+1}} \frac{\rho(N+1)}{1-\rho} = \frac{1}{1-\rho}. \end{aligned}$$

Summarizing, we obtain

$$\frac{1}{\frac{1}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}} \geq \pi_0 \geq (1-\rho).$$

In this example, the Lyapunov approach yields the exact solution.

6.3 Minimality, subdominance and stabililty

Both invariant measures and absorption (or hitting) probabilities are minimal solutions of an infinite system of linear equations. In many situations, minimal solutions are dominated by other solutions, in which case forward computation may become unstable. But our algorithms (backward recursions for $L_{\ldots}$ and $R_{\ldots}$ respectively and forward iterations for β_n and ψ_n respectively) are not susceptible to being influenced by dominant solutions: The approximants will be smaller than β_n and ψ_n respectively in any case. In chapter 8, we will deal more with dominance and subdominance, here we give two simple examples.

Example 6.3.1. Consider the $M/M/1$-queue with $\rho = \frac{\lambda}{\mu} < 1$ again, and let ψ be the invariant measure with $\psi_0 = 1$, that is $\psi_n = \rho^n$. Then ψ meets

$$\begin{aligned} -\psi_0\lambda + \psi_1\mu &= 0, \\ \psi_{n-1}\lambda - \psi_n(\lambda+\mu) + \psi_{n+1}\mu &= 0, \qquad n \in \mathbb{N}, \end{aligned}$$

and in principle, ψ could be computed by forward computation, that is, by

$$\psi_1 = \frac{\lambda}{\mu}\psi_0, \tag{6.3.1}$$

$$\psi_{n+1} = \frac{(\lambda+\mu)\psi_n - \lambda\psi_{n-1}}{\mu}, \qquad n \in \mathbb{N}. \tag{6.3.2}$$

Using doubles in `C++`, numerical computation for $\rho = \frac{1}{3}$ yields

n	exact solution ψ_n	forward computation	$\psi_n^{(N)}$ with $N = 100$
0	1	1	1
1	0.3333	0.333333	0.3333
2	0.1111	0.111111	0.1111
3	0.03704	0.03704	0.03704
4	0.01235	0.01235	0.01235
5	0.004115	0.004115	0.004115
6	0.001372	0.001372	0.001372
7	$4.572 \cdot 10^{-4}$	$4.572 \cdot 10^{-4}$	$4.572 \cdot 10^{-4}$
8	$1.524 \cdot 10^{-4}$	$1.524 \cdot 10^{-4}$	$1.524 \cdot 10^{-4}$
10	$1.694 \cdot 10^{-5}$	$1.694 \cdot 10^{-5}$	$1.694 \cdot 10^{-5}$
12	$1.882 \cdot 10^{-6}$	$1.882 \cdot 10^{-6}$	$1.882 \cdot 10^{-6}$
15	$6.969 \cdot 10^{-8}$	$6.969 \cdot 10^{-8}$	$6.969 \cdot 10^{-8}$
20	$2.868 \cdot 10^{-10}$	$2.868 \cdot 10^{-10}$	$2.868 \cdot 10^{-10}$
25	$1.180 \cdot 10^{-12}$	$1.180 \cdot 10^{-12}$	$1.180 \cdot 10^{-12}$
30	$4.857 \cdot 10^{-15}$	$4.870 \cdot 10^{-15}$	$4.857 \cdot 10^{-15}$
40	$8.225 \cdot 10^{-20}$	$1.312 \cdot 10^{-17}$	$8.225 \cdot 10^{-20}$
50	$1.393 \cdot 10^{-24}$	$1.304 \cdot 10^{-17}$	$1.393 \cdot 10^{-24}$
90	$1.146 \cdot 10^{-43}$	$1.304 \cdot 10^{-17}$	$1.146 \cdot 10^{-43}$
95	$4.715 \cdot 10^{-46}$	$1.304 \cdot 10^{-17}$	$4.709 \cdot 10^{-46}$
98	$1.746 \cdot 10^{-47}$	$1.304 \cdot 10^{-17}$	$1.682 \cdot 10^{-47}$
99	$5.821 \cdot 10^{-48}$	$1.304 \cdot 10^{-17}$	$5.174 \cdot 10^{-48}$
100	$1.940 \cdot 10^{-48}$	$1.304 \cdot 10^{-17}$	$1.294 \cdot 10^{-48}$

For the code, we refer to section D.1 in the appendix. Obviously, forward computing becomes unstable for large n. Since the dominant solution for (6.3.2) is given by $(1)_{n\in\mathbb{N}_0}$, forward computation yields an asymptotically constant solution. On the other hand, for $n = 0, \ldots, 90$, the continued fraction based solution is trustworthy, and if we would choose N larger, we could obtain good results for $n \geq 100$.

Example 6.3.2. In a second example, we demonstrate that the effect of instable forward computation can be much more dramatic. Consider the generator matrix

$$\mathbf{Q} = \begin{pmatrix} -1 & 1 & & & \\ \frac{a_0}{\gamma} & -\frac{a_0+1}{\gamma} & \frac{1}{\gamma} & & \\ & \frac{a_0}{\gamma^2} & -\frac{a_0+1}{\gamma^2} & \frac{1}{\gamma^2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with $a_0 > \gamma > 1$. Then we have positive recurrence, since the system $\psi\mathbf{Q} = 0$, that is

$$\begin{aligned} -\psi_0 + \psi_1 \frac{a_0}{\gamma} &= 0, \\ \psi_{n-1}\frac{1}{\gamma^{n-1}} - \psi_n \frac{a_0+1}{\gamma^n} + \psi_{n+1}\frac{a_0}{\gamma^{n+1}} &= 0, \qquad n \in \mathbb{N}, \end{aligned}$$

is solved by the (finite) invariant measure ψ with $\psi_n = \left(\frac{\gamma}{a_0}\right)^n$. Theoretically, ψ could be obtained by forward computation:

$$\psi_1 = \frac{\gamma\psi_0}{a_0}, \tag{6.3.3}$$

$$\psi_{n+1} = \frac{\psi_n(a_0+1)\gamma - \psi_{n-1}\gamma^2}{a_0}, \qquad n \in \mathbb{N}. \tag{6.3.4}$$

Here, for $\gamma = 9$ and $a_0 = 10$, numerical computation yields

n	exact solution ψ_n	forward computation	$\psi_n^{(N)}$ with $N = 100$
0	1	1	1
1	0.9	0.9	0.9
2	0.81	0.81	0.81
3	0.729	0.729	0.729
4	0.6561	0.6561	0.6561
5	0.5905	0.5905	0.5905
6	0.5314	0.5314	0.5314
7	0.4783	0.4783	0.4783
8	0.4305	0.4305	0.4305
10	0.3487	0.3487	0.3487
12	0.2824	0.2824	0.2824
15	0.2059	0.2066	0.2059
16	0.1853	0.1918	0.1853
17	0.1668	0.2250	0.1668
18	0.1501	0.6738	0.1501
19	0.1351	4.848	0.1351
20	0.1216	42.54	0.1216
25	0.07179	$2.505 \cdot 10^6$	0.07179
30	0.04239	$1.479 \cdot 10^{11}$	0.04239
40	0.01478	$5.157 \cdot 10^{20}$	0.01478
50	0.005154	$1.798 \cdot 10^{30}$	0.005154
90	$7.618 \cdot 10^{-5}$	$2.658 \cdot 10^{68}$	$7.618 \cdot 10^{-5}$
95	$4.498 \cdot 10^{-5}$	$1.569 \cdot 10^{73}$	$4.498 \cdot 10^{-5}$
98	$3.279 \cdot 10^{-5}$	$1.144 \cdot 10^{76}$	$3.276 \cdot 10^{-5}$
99	$2.951 \cdot 10^{-5}$	$1.030 \cdot 10^{77}$	$2.922 \cdot 10^{-5}$
100	$2.656 \cdot 10^{-5}$	$9.627 \cdot 10^{77}$	$2.391 \cdot 10^{-5}$

Again, for the code, we refer to section D.1 in the appendix. Here, a second solution for (6.3.4) is given by $(\gamma^n)_{n\in\mathbb{N}_0} = (9^n)_{n\in\mathbb{N}_0}$, and hence, forward computation yields a solution which asymptotically behaves like $c \cdot 9^n$. Again, the results of the continued fraction based computation are trustworthy for $n = 0, \ldots, 90$, and if we would choose N larger, we would obtain good results for $n \geq 100$, too.

Remark: By means of *uniformization* (see [Ste09, section 10.7.2]) the computation of ψ is equivalent to the computation in the introductory example in section 1.1: By setting $\mathbf{P} = \frac{1}{q}\mathbf{Q} + I$ with $q \geq \sup q_{ii}$, we obtain a stochastic matrix $\mathbf{P}$ with the same invariant

measures. For $a_0 = 10$ and $\gamma = 9$, we may choose $q = 2$, and obtain

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{10}{2\cdot 9} & 1 - \frac{11}{2\cdot 9} & \frac{1}{2\cdot 9} & & & \\ & \frac{10}{2\cdot 9^2} & 1 - \frac{11}{2\cdot 9^2} & \frac{1}{2\cdot 9^2} & & \\ & & \frac{10}{2\cdot 9^3} & 1 - \frac{11}{2\cdot 9^3} & \frac{10}{2\cdot 9^3} & \\ & & & \ddots & & \ddots \end{pmatrix}.$$

The reason for forward computation to fail is that the difference equation for $n \geq 1$ which is used for computing $\psi_2, \psi_3, \ldots$ has two linearly independent solutions: In the first example, these solutions are (ρ^n) and (1). In the second example, these solutions are $\left(\left(\frac{\gamma}{a_0}\right)^n\right)$ and (γ^n). Hence, the solutions (ρ^n) and $\left(\left(\frac{\gamma}{a_0}\right)^n\right)$ are dominated, and numerical computation will yield the dominant solutions.

On the other hand, the backward-forward method as suggested in Algorithm 6.2.1 is bounded by ψ_n. In section 8.1 we will discuss Miller's backward algorithm for solving difference equations. We will prove that – more or less – the gcf-based method and Miller's algorithm are equivalent. As we will point out in the discussion of Miller's algorithm, for difference equations of order ≥ 3, there is no guarantee that Miller's algorithm will compute the dominated solution. However, in the context of Markov chains, the gcf-based method, and hence, Miller's algorithm will compute the minimal non-negative solution. This solution may be dominated by another solution of some difference equation, and hence forward computation becomes unstable.

6.4 Literature review

As pointed out in section 5.4, lower block-Hessenberg matrices with periodic structure

$$\mathbf{Q} = \begin{pmatrix} B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & \\ B_2 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

were analyzed by Neuts [Neu81], invariant measures and invariant distributions were characterized. In section 5.4, we already mentioned that $R_{n-1,n,n} = R$ does not depend on n, and hence, $\psi_n = \psi_0 R^n$ for any invariant measure ψ.

In particular, for *level-independent quasi-birth-death processes*, that is

$$\mathbf{Q} = \begin{pmatrix} B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

Neuts characterized invariant measures by $\psi_n = \psi_0 R^n$, where R is the minimal non-negative solution of

$$A_0 + RA_1 + R^2 A_2 = 0.$$

Neuts already considered cases of quasi-birth-death processes where $q_{n-1,n} = A_0$, $q_{nn} = A_1$ and $q_{n+1,n} = A_2$ only hold for $n \geq n_0$ with some $n_0 \in \mathbb{N}$ (instead of $n \geq 1$). Bright and Taylor [BT95] were the first authors considering *level-dependent quasi-birth-death processes*, that is, there is no such n_0. In the discussion of Algorithm 6.2.1, we pointed out that our method is quite similar to that suggested by Bright and Taylor, and similar algorithms were discussed in [Han99], [BS10], [PD+10], applications can be found in [Her08], [BS12a] and [BS16].

Quasi-birth-death processes with catastrophes, that is, block-matrices

$$\mathbf{Q} = \begin{pmatrix} q_{00} & q_{01} & & & & \\ q_{10} & q_{11} & q_{12} & & & \\ q_{20} & q_{21} & q_{22} & q_{23} & & \\ q_{30} & & q_{32} & q_{33} & q_{34} & \\ \vdots & & & \ddots & \ddots & \ddots \end{pmatrix}$$

were discussed in [BS12b]. These processes are special cases of lower block-Hessenberg matrices, but the special structure made it easier to compute $K_0^{(N)}$.

For quasi-birth-death processes, the Horner-type scheme for the direct computation of ψf was explained in [BS13], applications are given in [Hu+15] and [BS17].

To the knowledge of the author, algorithms for computing ψf for upper or lower block-Hessenberg matrices are not available so far. Furthermore, up to now, the direct computation of (total) hitting probabilities or mean hitting times have not been studied. However, we have to admit that the algorithms are quite similar to those for computing ψf.

Lyapunov functions have been used before for finding error estimates in the context of invariant measures of Markov chains, but as pointed out in section 5.4, in a slightly different way. To the knowledge of the author, the algorithms which simultaneously lead to lower and upper bounds are completely new.

Part II

Generalized continued fractions defined by sequences of numerators and denominators

Chapter 7

GCFs generated by upper Hessenberg matrices

Consider the Nth approximant

$$K_0^{(N)} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\ddots \cfrac{}{b_2 + \cfrac{}{b_{N-1} + \cfrac{a_N}{b_N}}}}}$$

of the infinite non-generalized continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}$$

with $b_n, a_n \in \mathbb{C}$. The 'traditional analysis' of convergence of continued fractions uses that $K_0^{(N)} = \frac{A_N}{B_N}$, where the sequences $(A_n)_{n\geq -1}$ and $(B_n)_{n\in\mathbb{N}_0}$ are defined by $B_{-1} = 0$, $B_0 = 1$, $A_{-1} = I$, $A_0 = b_0$ and

$$\begin{aligned} A_n &= A_{n-1}b_n + A_{n-2}a_n, \qquad n \geq 1, \\ B_n &= B_{n-1}b_n + B_{n-2}a_n, \qquad n \geq 1, \end{aligned}$$

that is, both numerator and denominator meet the same (second-order) recurrence relation, see [Per54]. Hence, it does not surprise that often, continued fractions are defined by this type of recurrence scheme. Furthermore, many generalizations (of the domain or of the recurrence scheme) can be found in the literature, see section 7.4 at the end of this chapter.

Our definition of gcfs may seem to be quite different. However, this is not true. In this second part of the thesis, we will consider the special case of gcfs generated by upper Hessenberg matrices with $c_{n+1,n} \in \mathcal{R}^*$. This is due to several reasons:

- It will turn out that the approximants of such gcfs still have a representation as ordinary fractions where numerator and denominator meet a forward recurrence relation, see section 7.1.
- This representation allows us to give an alternative proof for Pringsheim-type convergence criteria in which we do not make any use of S-series or probabilistic interpretations of the approximants, see section 7.3.

- With the recurrence scheme for numerator and denominator, we will be able to prove that our definition of gcfs include all generalizations of continued fractions which we present in the literature review at the end of this chapter.
- Without stating this fact explicitly, we have used *adjoint* systems in chapter 5: In Theorem 5.2.1, we stated that the hitting probabilities solve $\mathbf{H}x = 0$, and for the error estimate, we used a solution of $\mu\mathbf{H} \geq 0$. Similarly, in Theorem 5.2.3, we saw that the invariant distribution solves $\psi\mathbf{H} = 0$, and for the error estimate, we used a solution of $\mathbf{H}g \geq 0$. In chapter 8, we will develop stronger relationships for upper Hessenberg matrices.
- For upper Hessenberg matrices, up to the initial condition, $\mathbf{H}x = 0$ reads as

$$c_{n,n-1}x_{n-1} = b_n x_n + \sum_{m=n+1}^{\infty} a_{nm}x_m, \qquad n \in \mathbb{N},.$$

  As before, we refer to this type of equation as *sum equation*, see [Per20; Per21]. In particular, for $c_{n+1,n} = -\alpha_0$, $b_n = \alpha_1$ and $a_{nm} = \alpha_{m+1-n}$, we obtain

$$\sum_{k=0}^{\infty} \alpha_k x_{n-1+k} = 0,$$

  and candidates for solving this equation are given by $x_k = z^k$, where z is a solution of $0 = F(z) = \sum_{k=0}^{\infty} \alpha_k z^k$, see [Per20]. Since gcf-based solutions are related to minimal solutions, there is a justified hope that gcfs are appropriate for characterizing minimal roots of analytic functions. We will discuss this topic in chapter 9.

7.1 Representation by numerators and denominators

We directly introduce the representation for $K_0^{(N)}$ and $L_{0,1,n}^{(N)}$ as 'ordinary fractions'. Recall that for upper Hessenberg matrices $\mathbf{H}$, we have

$$\mathbf{H} = \begin{pmatrix} b_0 & a_{01} & a_{02} & a_{03} & \cdots \\ -c_{10} & b_1 & a_{12} & a_{23} & \cdots \\ & -c_{21} & b_2 & a_{23} & \cdots \\ & & -c_{32} & b_3 & \cdots \\ & & & -c_{43} & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Theorem 7.1.1. *Let $\mathbf{H}$ be an upper Hessenberg matrix with $c_{n+1,n} \in \mathcal{R}^*$ for all $n \in \mathbb{N}_0$, and let $\operatorname{gcf}(\mathbf{H})$, $K_0^{(N)}$ and $L_{n,1,0}^{(N)}$ be defined according to Definition 2.3.1. Define the sequences $B^{(\ell)} = \left(B_n^{(\ell)}\right)_{n\in\mathbb{N}_0}$ for $\ell \in \mathbb{N}_0 \cup \{-1\}$ as solutions of the recursion*

$$X_n c_{n+1,n} = X_{n-1}b_n + \sum_{m=0}^{n-1} X_{m-1}a_{mn}, \qquad n > \ell, \tag{7.1.1}$$

subject to the initial conditions

$$B_n^{(\ell)} = 0, \quad n < \ell, \qquad B_\ell^{(\ell)} c_{\ell+1,\ell} = I, \quad \ell \in \mathbb{N}_0, \qquad B_{-1}^{(-1)} = I, \tag{7.1.2}$$

and put

$$A_n = B_n^{(-1)} \qquad \textit{and} \qquad B_n = B_n^{(0)}.$$

If $K_0^{(N)}$ is well-defined for some $N \in \mathbb{N}_0$, B_N^{-1} exists and $K_0^{(N)} = A_N B_N^{-1}$. If $L_{\ell,1,0}^{(N)}$ is well-defined for some $N \in \mathbb{N}$, B_N^{-1} exists and $L_{\ell,1,0}^{(N)} = B_N^{(\ell)} B_N^{-1}$.

Proof. First, we prove the statement for $K_0^{(N)}$ by induction with respect to N.

For $N = 0$, we have $K_0^{(0)} = b_0$, $A_0 = b_0 c_{10}^{-1}$ and $B_0 = c_{10}^{-1}$, and the statement is obviously true. Similarly, for $N = 1$, we have $K_0^{(1)} = b_0 + a_{01} b_1^{-1} c_{10}$, $A_1 = b_0 c_{10}^{-1} b_1 c_{21}^{-1} + a_{01} c_{21}^{-1}$ and $B_1 = c_{10}^{-1} b_1 c_{21}^{-1}$, and again, the statement is true.

Now assume that the statement is true for the $(N-1)$st approximants of all gcfs generated by upper Hessenberg-matrices. Then,

$$\tilde{K}_0^{(N-1)} = \tilde{A}_{N-1} \tilde{B}_{N-1}^{-1},$$

where $\tilde{K}_0^{(N-1)}$, $\tilde{A}_{N-1}$ and $\tilde{B}_{N-1}$ are constructed using the coefficients $\tilde{b}_n$, $\tilde{a}_{mn}$ and $\tilde{c}_{n,n-1}$, where $\tilde{b}_n = b_n$ for $n \le N-2$, $\tilde{a}_{mn} = a_{mn}$ for $m < n \le N-2$, $\tilde{c}_{n,n-1} = c_{n,n-1}$ for $n \le N-1$ and

$$\tilde{b}_{N-1} = b_{N-1} + a_{N-1,N} b_N^{-1} c_{N,N-1}, \quad \tilde{a}_{m,N-1} = a_{m,N-1} + a_{mN} b_N^{-1} c_{N,N-1}, \quad m \le N-2.$$

By choice of $\tilde{b}_{N-1}$, we have $\tilde{K}_{N-1}^{(N-1)} = K_{N-1}^{(N)}$, and the choice of $\tilde{a}_{m,N-1}$ yields iteratively

$$\begin{aligned}
\tilde{K}_n^{(N-1)} &= \tilde{b}_n + \sum_{m=n+1}^{N-1} \tilde{a}_{nm} \prod_{n+1}^{\ell=m} \left(\left(\tilde{K}_\ell^{(N-1)} \right)^{-1} \tilde{c}_{\ell,\ell-1} \right) \\
&= b_n + \sum_{m=n+1}^{N-1} a_{nm} \prod_{n+1}^{\ell=m} \left(\left(K_\ell^{(N)} \right)^{-1} c_{\ell,\ell-1} \right) \\
&\qquad + a_{n,N} b_N^{-1} c_{N,N-1} \prod_{n+1}^{\ell=N-1} \left(\left(K_\ell^{(N)} \right)^{-1} c_{\ell,\ell-1} \right) \\
&= b_n + \sum_{m=n+1}^{N} a_{nm} \prod_{n+1}^{\ell=m} \left(\left(K_\ell^{(N)} \right)^{-1} c_{\ell,\ell-1} \right) = K_n^{(N)}
\end{aligned}$$

for alle $n \le N-2$. In particular, we have

$$\begin{aligned}
K_0^{(N)} &= \tilde{K}_0^{(N-1)} = \tilde{A}_{N-1} \tilde{B}_{N-1}^{-1} = \tilde{A}_{N-1} \tilde{c}_{N,N-1} \left(\tilde{B}_{N-1} c_{N,N-1} \right)^{-1} \\
&= \left(\tilde{A}_{N-2} \tilde{b}_{N-1} + \sum_{m=0}^{N-2} \tilde{A}_{m-1} \tilde{a}_{m,N-1} \right) \left(\tilde{B}_{N-2} \tilde{b}_{N-1} + \sum_{m=0}^{N-2} \tilde{B}_{m-1} \tilde{a}_{m,N-1} \right)^{-1},
\end{aligned}$$

and since $\tilde{A}_m = A_m$ and $\tilde{B}_m = B_m$ for all $m \leq N-2$, we finally obtain

$$\begin{aligned}
K_0^{(N)} &= \left(A_{N-2}b_{N-1} + \sum_{m=0}^{N-2} A_{m-1}a_{m,N-1} + A_{N-2}a_{N-1,N}b_N^{-1}c_{N,N-1} + \sum_{m=0}^{N-2} A_{m-1}a_{m,N}b_N^{-1}c_{N,N-1} \right) \\
&\quad \cdot \left(B_{N-2}b_{N-1} + \sum_{m=0}^{N-2} B_m a_{m,N-1} + B_{N-2}^{(0)}a_{N-1,N}b_N^{-1}c_{N,N-1} + \sum_{m=0}^{N-2} B_{m-1}a_{m,N}b_N^{-1}c_{N,N-1} \right)^{-1} \\
&= \left(A_{N-1}c_{N,N-1} + A_{N-2}a_{N-1,N}b_N^{-1}c_{N,N-1} + \sum_{m=0}^{N-2} A_{m-1}a_{m,N}b_N^{-1}c_{N,N-1} \right) \\
&\quad \cdot \left(B_{N-1}c_{N,N-1} + B_{N-2}a_{N-1,N}b_N^{-1}c_{N,N-1} + \sum_{m=0}^{N-2} B_{m-1}a_{m,N}b_N^{-1}c_{N,N-1} \right)^{-1} \\
&= \left(A_{N-1}b_N + \sum_{m=0}^{N-1} A_{m-1}a_{mN} \right) \left(B_{N-1}b_N + \sum_{m=0}^{N-1} B_{m-1}a_{mN} \right)^{-1} \\
&= A_N c_{N+1,N} \left(B_N c_{N+1,N} \right)^{-1} = A_N B_N^{-1}.
\end{aligned}$$

The statement for $L_{\ell,1,0}^{(N)}$ can be obtained from setting $b_0 = 0$ and $a_{0m} = \delta_{m\ell} I$. Obviously, this assumption has no influence on $L_{\ell,1,0}^{(N)}$ but guarantees $K_0^{(N)} = L_{\ell,1,0}^{(N)}$. On the other hand, it yields $A_N = 0$ for $N = 0, \ldots, n-1$, $A_\ell c_{\ell+1,\ell} = A_{-1} = I$ and

$$A_N c_{N+1,N} = A_{N-1}b_N + \sum_{m=\ell+1}^{N-1} A_{m-1}a_{mN},$$

that is $A_N = B_N^{(\ell)}$. □

Theorem 7.1.1 suggests an alternative definition for gcfs built up by upper Hessenberg-matrices with $c_{n+1,n} \in \mathcal{R}^*$.

Definition 7.1.1. Let $\mathbf{H}$ be an $\mathcal{R}$-valued infinite upper Hessenberg-matrix

$$\mathbf{H} = \begin{pmatrix}
b_0 & a_{01} & a_{02} & a_{03} & \cdots \\
-c_{10} & b_1 & a_{12} & a_{13} & \cdots \\
 & -c_{21} & b_2 & a_{23} & \cdots \\
 & & -c_{32} & b_3 & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}$$

with $c_{n+1,n} \in \mathcal{R}^*$ for all $n \in \mathbb{N}_0$. Define $A_n, B_n^{(\ell)}$ for $n, \ell \in \mathbb{N}_0$ as in Theorem 7.1.1, and let $B_n = B_n^{(0)} \in \mathcal{R}^*$ for almost all $n \in \mathbb{N}$.

- If
$$K = \lim_{N\to\infty} A_N B_N^{-1}$$
exists, K is said to be a *convergent generalized continued fraction*, abbreviated by
$$K = \operatorname{gcf}(\mathbf{H}).$$
$K_0^{(N)} = A_N B_N^{-1}$ is referred to as *Nth approximant for* $\operatorname{gcf}(\mathbf{H})$.

- We define

$$w_n^{(N)}(w_0, \mathbf{H}) := B_N^{(n)} B_N^{-1} w_0 \text{ and } w^{(N)}(w_0, \mathbf{H}) = \left(w_n^{(N)}(w_0, \mathbf{H}) \right)_{n \in \mathbb{N}_0}.$$

  If the limits exist, we write

$$w_n(w_0, \mathbf{H}) = \lim_{N \to \infty} w_n^{(N)}(w_0, \mathbf{H}) \text{ and } w(w_0, \mathbf{H}) = \left(w_n(w_0, \mathbf{H}) \right)_{n \in \mathbb{N}_0}.$$

In what follows, we will use this definition of gcfs whenever it is applicable. Note that

- for upper Hessenberg matrices with $c_{n+1,n} \in \mathcal{R}^*$, Definition 7.1.1 is slightly more general than Definition 2.3.1. In order to demonstrate this fact, consider $\mathcal{R} = \mathbb{C}$ and a tridiagonal matrix $\mathbf{H}$ with $b_0 = b_1 = b_2 = b_3 = 1$, $a_{01} = a_{12} = 1$, $a_{23} = -1$ and $c_{10} = c_{21} = c_{32} = c_{34} = 1$. Then $K_0^{(3)}$ is the ordinary continued fraction

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{-1}{1}}}$$

  in $\mathbb{C}$, and since $K_2^{(3)} = 0$, $K_0^{(3)}$ is not well-defined according to Definition 2.3.1. On the other hand, $A_{-1} = A_0 = 1$, $A_1 = 2$, $A_2 = 3$, $A_4 = 1$ and $B_0 = B_1 = 1$, $B_2 = 2$, $B_3 = 1 \neq 0$, that is, $K_0^{(3)} = \frac{A_3}{B_3} = 1$ is well-defined according to Definition 7.1.1. On the other hand, Theorem 7.1.1 guarantees that whenever Definition 2.3.1 yields a well-defined gcf, so does Definition 7.1.1.

- with a simple equivalence transformation (see section 3.1 or 7.2), we can replace $c_{n+1,n} \in \mathcal{R}^*$ by I, so we could assume $c_{n+1,n} = I$ without loss of generality. We will not use this consideration for means of comparison with results from the first part.

7.2 Equivalence transformations revisited

As for gcfs defined according to Definition 2.3.1, we can apply equivalence transformations to gcfs defined according to Definition 7.1.1, that is, we consider $\tilde{c}_{n+1,n} = \lambda_{n+1} c_{n+1,n} \rho_n^{-1}$, $\tilde{b}_n = \lambda_n b_n \rho_n^{-1}$ and $\tilde{a}_{mn} = \lambda_m a_{mn} \rho_n^{-1}$. Then

$$\tilde{A}_N = \lambda_0 A_N \lambda_{N+1}^{-1}, \ \tilde{B}_N^{(\ell)} = \rho_\ell B_N^{(\ell)} \lambda_{N+1}^{-1},$$

for $N \geq -1$ and $N \geq \ell$ respectively follow with simple inductions. Hence, we obtain

$$\begin{aligned} \tilde{A}_N \tilde{B}_N^{-1} &= \lambda_0 A_N B_N^{-1} \rho_0^{-1} \qquad \text{and} \\ \tilde{B}_N^{(\ell)} \tilde{B}_N^{-1} &= \rho_\ell B_N^{(\ell)} B_N^{-1} \rho_0^{-1}, \end{aligned}$$

and thus

Theorem 7.2.1. *Let $\mathbf{H}$ be an upper Hessenberg matrix with $c_{n+1,n} \in \mathcal{R}^*$ for all $n \in \mathbb{N}$, let $(\lambda_n)_{n \in \mathbb{N}_0}$ and $(\rho_n)_{n \in \mathbb{N}_0}$ be $\mathcal{R}^*$-valued sequences, and define $\tilde{\mathbf{H}}$ by $\tilde{c}_{n+1,n} = \lambda_{n+1} c_{n+1,n} \rho_n^{-1}$, $\tilde{b}_n = \lambda_n c_n \rho_n^{-1}$ and $\tilde{a}_{mn} = \lambda_m a_{mn} \rho_n^{-1}$.*

- $\operatorname{gcf}(\mathbf{H})$ *converges if and only if* $\operatorname{gcf}\left(\tilde{\mathbf{H}}\right)$ *converges. In case of convergence, we have* $\operatorname{gcf}\left(\tilde{\mathbf{H}}\right) = \lambda_0 \operatorname{gcf}(\mathbf{H})\,\rho_0^{-1}$.

- $w_n(w_0, \mathbf{H})$ *converges if and only if* $w_n\left(\rho_0 w_0, \tilde{\mathbf{H}}\right)$ *converges. In case of convergence, we have* $w_n\left(\rho_0 w_0, \tilde{\mathbf{H}}\right) = \rho_n w_n(w_0, \mathbf{H})$.

7.3 Pringsheim-type convergence criteria: Alternative proof

In this section, we aim at finding an 'elementary proof' for Pringsheim-type convergence criteria for gcfs defined according to Definition 7.1.1. In principle, we extend the ideas of [Bau13], where a Pringsheim-type criterion for non-generalized continued fractions in Banach algebras was proved.

7.3.1 The difference between approximants

A basic result for continued fractions in $\mathbb{C}$ is the *Euler-Minding formula*

$$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1} a_1 \cdot a_2 \cdot \ldots \cdot a_n,$$

see [Per54]. Equivalently, we have

$$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = (-1)^{n-1} \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{B_{n-1} B_n}, \tag{7.3.1}$$

and on the left-hand side, we have the difference between two approximants of the continued fraction. This result is often used when proving convergence criteria.

Now we turn to find an analogous result for gcfs generated by upper Hessenberg matrices. In this section, we use

$$D_n = K_0^{(n)} - K_0^{(n-1)} = A_n B_n^{-1} - A_{n-1} B_{n-1}^{-1}$$

if $B_n, B_{n-1} \in \mathcal{R}^*$. Obviously, we can write

$$\operatorname{gcf}(\mathbf{H}) = A_{n_0} B_{n_0}^{-1} + \sum_{n=n_0+1}^{\infty} D_n,$$

for any $n_0 \in \mathbb{N}_0$, and in particular, if

$$\sum_{n=n_0+1}^{\infty} ||D_n||$$

converges for some $n_0 \in \mathbb{N}_0$, so does $\operatorname{gcf}(\mathbf{H})$. In the scalar case, this kind of convergence of a continued fraction is referred to as *absolute convergence*, see [Per57].

The appropriate generalization of (7.3.1) is

Lemma 7.3.1. *Let $B_n \in \mathcal{R}^*$ for all $n \in \mathbb{N}$. Then $D_1 = a_{01}b_1^{-1}c_{10}$ and for $n \geq 2$, we have*

$$D_n = -\left(\sum_{m=1}^{n-1}\sum_{\ell=m}^{n-1} D_\ell B_{m-1}a_{mn} - a_{0n}\right)c_{n+1,n}^{-1}B_n^{-1}. \tag{7.3.2}$$

Proof. According to the definitions of (A_n) and (B_n) respectively, we have

$$A_n c_{n+1,n} = A_{n-1}b_n + \sum_{m=0}^{n-1} A_{m-1}a_{mn}, \qquad n \in \mathbb{N}_0, \qquad A_{-1} = I, \tag{7.3.3}$$

$$B_n c_{n+1,n} = B_{n-1}b_n + \sum_{m=1}^{n-1} B_{m-1}a_{mn}, \qquad n \in \mathbb{N}, \qquad B_0 = I. \tag{7.3.4}$$

The definition $D_n = A_nB_n^{-1} - A_{n-1}B_{n-1}^{-1}$ yields

$$D_1 = A_1B_1^{-1} - A_0B_0^{-1} = (b_0c_{10}^{-1}b_1 + a_{01})c_{21}^{-1}\left(\left(c_{10}^{-1}b_1\right)c_{21}^{-1}\right)^{-1} - b_0c_{10}^{-1}c_{10} = a_{01}b_1^{-1}c_{10}$$

and

$$\begin{aligned} D_n &= (A_n - A_{n-1}B_{n-1}^{-1}B_n)B_n^{-1} = (A_nc_{n+1,n} - A_{n-1}B_{n-1}^{-1}B_nc_{n+1,n})(B_nc_{n+1,n})^{-1} \\ &= \left(\sum_{m=0}^{n-1} A_{m-1}a_{mn} - \sum_{m=1}^{n-1} A_{n-1}B_{n-1}^{-1}B_{m-1}a_{mn}\right)c_{n+1,n}^{-1}B_n^{-1} \\ &= -\left(\sum_{m=1}^{n-1}\left(A_{n-1}B_{n-1}^{-1} - A_{m-1}B_{m-1}^{-1}\right)B_{m-1}a_{mn} - a_{0n}\right)c_{n+1,n}^{-1}B_n^{-1}. \end{aligned}$$

□

For non-generalized continued fractions, that is $c_{n+1,n} = I$ and $a_{mn} = 0$ for $n-m > 1$, (7.3.2) simplifies to

$$D_n = -D_{n-1}B_{n-2}a_{n-1,n}B_n^{-1}. \tag{7.3.5}$$

This result was used in [Bau13]. For $\mathcal{R} = \mathbb{C}$, by means of commutativity, (7.3.5) is equivalent to the Euler-Minding formula (7.3.1).

Next, we prove a result concerning the differences D_n for the case that a Pringsheim-type condition holds.

Lemma 7.3.2. *Let $b_n \in \mathcal{R}^*$ for all $n \in \mathbb{N}$ and let*

$$\left|\left|c_{n+1,n}b_n^{-1}\right|\right| + \sum_{m=0}^{n-1}\left|\left|a_{mn}b_n^{-1}\right|\right| \leq 1 \tag{7.3.6}$$

hold for all $n \in \mathbb{N}$. Then $B_n \in \mathcal{R}^*$ for all $n \in \mathbb{N}$ and

$$\left|\left|B_n^{-1}B_{n-1}\right|\right| + ||D_nB_{n-1}|| \leq 1 \tag{7.3.7}$$

holds for all $n \in \mathbb{N}$.

Proof. For $n = 1$, we have $B_0 = c_{10}^{-1}$, $B_1^{-1}B_0 = c_{21}b_1^{-1}$, and

$$\left|\left|B_1^{-1}B_0\right|\right| + ||D_1B_0|| = \left|\left|c_{21}b_1^{-1}\right|\right| + \left|\left|a_{01}b_1^{-1}\right|\right| \leq 1$$

is an immediate consequence of (7.3.6).
Now assume $\left|\left|B_k^{-1}B_{k-1}\right|\right| + ||D_kB_{k-1}|| \leq 1$ for all $k = 1, \ldots, n-1$. From (7.3.4), we obtain

$$\begin{aligned} B_{n-1}^{-1}B_nc_{n+1,n}b_n^{-1} &= I + \sum_{m=1}^{n-1} B_{n-1}^{-1}B_{m-1}a_{mn}b_n^{-1} \\ &= I + \sum_{m=1}^{n-1}\left(\prod_{m}^{k=n-1} B_k^{-1}B_{k-1}\right)a_{mn}b_n^{-1}. \end{aligned}$$

The induction hypothesis yields $\left|\left|B_k^{-1}B_{k-1}\right|\right| \leq 1$ for $k = m, \ldots, n-1$, the Pringsheim-condition (7.3.6) implies $\sum\limits_{m=1}^{n-1}\left|\left|a_{mn}b_n^{-1}\right|\right| \leq 1 - \left|\left|c_{n+1,n}b_n^{-1}\right|\right| < 1$, and together we have

$$\left|\left|\sum_{m=1}^{n-1}\left(\prod_{m}^{k=n-1} B_k^{-1}B_{k-1}\right)a_{mn}b_n^{-1}\right|\right| < 1,$$

and thus, according to Lemma A.2.1, $\left(B_{n-1}^{-1}B_nc_{n+1,n}b_n^{-1}\right)^{-1}$ and hence B_n^{-1} exist. Furthermore, for any $q \in \mathcal{R}$, Lemma A.2.1 states

$$\begin{aligned} \left|\left|q\left(B_{n-1}^{-1}B_nc_{n+1,n}b_n^{-1}\right)^{-1}\right|\right| &\leq \frac{||q||}{1 - \left|\left|\sum\limits_{m=1}^{n-1} B_{n-1}^{-1}B_{m-1}a_{mn}b_n^{-1}\right|\right|} \\ &\leq \frac{||q||}{1 - \sum\limits_{m=1}^{n-1}\left|\left|B_{n-1}^{-1}B_{m-1}\right|\right| \cdot \left|\left|a_{mn}b_n^{-1}\right|\right|}. \end{aligned}$$

For $q = c_{n+1,n}b_n^{-1}$, we obtain

$$\begin{aligned} \left|\left|B_n^{-1}B_{n-1}\right|\right| &= \left|\left|c_{n+1,n}b_n^{-1}\left(B_{n-1}^{-1}B_nb_n^{-1}\right)^{-1}\right|\right| \\ &\leq \frac{\left|\left|c_{n+1,n}b_n^{-1}\right|\right|}{1 - \sum\limits_{m=1}^{n-1}\left|\left|B_{n-1}^{-1}B_{m-1}\right|\right| \cdot \left|\left|a_{mn}b_n^{-1}\right|\right|}, \end{aligned}$$

and for $q = D_nB_nc_{n+1,n}b_n^{-1}$, by means of (7.3.2), we obtain

$$\begin{aligned} ||D_nB_{n-1}|| &= \left|\left|\left(\sum_{m=1}^{n-1}\sum_{k=m}^{n-1} D_kB_{m-1}a_{mn}b_n^{-1} - a_{0n}b_n^{-1}\right)\left(B_{n-1}^{-1}B_nb_n^{-1}\right)^{-1}\right|\right| \\ &\leq \frac{\sum\limits_{m=1}^{n-1}\sum\limits_{k=m}^{n-1}||D_kB_{m-1}|| \cdot \left|\left|a_{mn}b_n^{-1}\right|\right| + \left|\left|a_{0n}b_n^{-1}\right|\right|}{1 - \sum\limits_{m=1}^{n-1}\left|\left|B_{n-1}^{-1}B_{m-1}\right|\right| \cdot \left|\left|a_{mn}b_n^{-1}\right|\right|}. \end{aligned}$$

Using the induction hypothesis, we derive

$$\begin{aligned}
\sum_{k=m}^{n-1} ||D_k B_{m-1}|| &\leq ||D_m B_{m-1}|| + \sum_{k=m+1}^{n-1} ||D_k B_{k-1}|| \cdot \left|\left|B_{k-1}^{-1} B_{m-1}\right|\right| \\
&\leq 1 - \left|\left|B_m^{-1} B_{m-1}\right|\right| + \sum_{k=m+1}^{n-1} \left(1 - \left|\left|B_k^{-1} B_{k-1}\right|\right|\right) \cdot \left|\left|B_{k-1}^{-1} B_{m-1}\right|\right| \\
&\leq 1 - \left|\left|B_m^{-1} B_{m-1}\right|\right| + \sum_{k=m+1}^{n-1} \left(\left|\left|B_{k-1}^{-1} B_{m-1}\right|\right| - \left|\left|B_k^{-1} B_{m-1}\right|\right|\right) \\
&= 1 - \left|\left|B_{n-1}^{-1} B_{m-1}\right|\right|,
\end{aligned}$$

and finally, by means of (7.3.6), we obtain

$$\begin{aligned}
& \left|\left|B_n^{-1} B_{n-1}\right|\right| + ||D_n B_{n-1}|| \\
\leq\ & \frac{\left|\left|c_{n+1,n} b_n^{-1}\right|\right| + \sum_{m=1}^{n-1} \left(1 - \left|\left|B_{n-1}^{-1} B_{m-1}\right|\right|\right) \cdot \left|\left|a_{mn} b_n^{-1}\right|\right| + \left|\left|a_{0n} b_n^{-1}\right|\right|}{1 - \sum_{m=1}^{n-1} \left|\left|B_{n-1}^{-1} B_{m-1}\right|\right| \cdot \left|\left|a_{mn} b_n^{-1}\right|\right|} \\
=\ & \frac{\left|\left|c_{n+1,n} b_n^{-1}\right|\right| + \sum_{m=0}^{n-1} \left|\left|a_{mn} b_n^{-1}\right|\right| - \sum_{m=1}^{n-1} \left|\left|B_{n-1}^{-1} B_{m-1}\right|\right| \cdot \left|\left|a_{mn} b_n^{-1}\right|\right|}{1 - \sum_{m=1}^{n-1} \left|\left|B_{n-1}^{-1} B_{m-1}\right|\right| \cdot \left|\left|a_{mn} b_n^{-1}\right|\right|} \\
\leq\ & \frac{1 - \sum_{m=1}^{n-1} \left|\left|B_{n-1}^{-1} B_{m-1}\right|\right| \cdot \left|\left|a_{mn} b_n^{-1}\right|\right|}{1 - \sum_{m=1}^{n-1} \left|\left|B_{n-1}^{-1} B_{m-1}\right|\right| \cdot \left|\left|a_{mn} b_n^{-1}\right|\right|} = 1.
\end{aligned}$$

□

7.3.2 Pringsheim-type convergence criteria for gcf (**H**)

In [Bau13], the statement of Lemma 7.3.2, that is (7.3.7), was proved for non-generalized continued fractions satisfying $\left|\left|b_n^{-1}\right|\right| + \left|\left|a_n b_n^{-1}\right|\right| \leq 1$. Using this result, a Pringsheim-type criterion was obtained easily. Since (7.3.7) is still valid in the general setting considered here, we can derive a Pringsheim-type criterion in the same manner.

Theorem 7.3.1. *Let $b_n^{-1} \in \mathcal{R}^*$ for all $n \in \mathbb{N}_0$ and let* (7.3.6) *hold for all $n \in \mathbb{N}$. Then* gcf (**H**) *converges and*

$$\left|\left|\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right|\right| \leq \left|\left|B_N^{-1}\right|\right|.$$

holds for all approximants $K_0^{(N)} = A_N B_N^{-1}$. In particular,

$$||\operatorname{gcf}(\mathbf{H}) - b_0|| \leq ||c_{10}||.$$

Proof. From Lemma 7.3.2, we have

$$\left|\left|B_n^{-1}B_{n-1}\right|\right| + ||D_nB_{n-1}|| \leq 1.$$

Hence,

$$\begin{aligned} ||D_n|| &\leq ||D_nB_{n-1}|| \cdot \left|\left|B_{n-1}^{-1}\right|\right| \leq \left(1 - \left|\left|B_n^{-1}B_{n-1}\right|\right|\right) \cdot \left|\left|B_{n-1}^{-1}\right|\right| \\ &\leq \left|\left|B_{n-1}^{-1}\right|\right| - \left|\left|B_n^{-1}\right|\right| \end{aligned}$$

for all $n \in \mathbb{N}$. Trivially, $||D_n|| \geq 0$, and thus, $\left|\left|B_n^{-1}\right|\right|$ decreases monotonically. Since $\left|\left|B_0^{-1}\right|\right| = ||c_{10}||$ and $\left|\left|B_n^{-1}\right|\right| \geq 0$, $\left|\left|B_n^{-1}\right|\right|$ converges to some $\beta \in [0, ||c_{10}||]$. Therefore,

$$\sum_{n=1}^{\infty} ||D_n|| \leq \sum_{n=1}^{\infty} \left(\left|\left|B_{n-1}^{-1}\right|\right| - \left|\left|B_n^{-1}\right|\right|\right) = ||c_{10}|| - \beta \in [0, ||c_{10}||].$$

For the errors, we then obtain

$$\left|\left|K - K_0^{(N)}\right|\right| = \left|\left|\sum_{n=N+1}^{\infty} D_n\right|\right| \leq \sum_{n=N+1}^{\infty} ||D_n|| \leq \left|\left|B_N^{-1}\right|\right| - \beta \leq \left|\left|B_N^{-1}\right|\right|.$$

□

Obviously, if the inequality in (7.3.6) turns out to be strict for some n_0, the inequality in (7.3.7) is strict for $n \geq n_0$. By applying equivalence transformations, we obtain

Theorem 7.3.2. *Let $b_n \in \mathcal{R}^*$ for all $n \in \mathbb{N}$, let $(\lambda_n)_{n\in\mathbb{N}_0}$ be a $\mathcal{R}^*$-valued sequence, and let*

$$\left|\left|\lambda_{n+1}c_{n+1,n}b_n^{-1}\lambda_n^{-1}\right|\right| + \sum_{m=0}^{n-1} \left|\left|\lambda_m a_{mn} b_n^{-1}\lambda_n^{-1}\right|\right| \leq 1 \tag{7.3.8}$$

hold for all $n \in \mathbb{N}$. Then $K = \operatorname{gcf}(\mathbf{H})$ converges and

$$\left|\left|\lambda_0 \left(K - K_0^{(N)}\right) \rho_0^{-1}\right|\right| \leq \left|\left|\lambda_{N+1}B_N^{-1}\rho_0^{-1}\right|\right|, \qquad n \in \mathbb{N}_0, \tag{7.3.9}$$

in particular

$$\left|\left|\lambda_0(K - b_0)\rho_0^{-1}\right|\right| \leq \left|\left|\lambda_1 c_{10}\rho_0^{-1}\right|\right| \tag{7.3.10}$$

holds for any $\rho_0 \in \mathcal{R}^*$. If the inequality (7.3.8) is strict for at least one $n \in \mathbb{N}$, we have strict inequality in (7.3.10).

Proof. According to Theorem 7.2.1, we put

$$\tilde{b}_n = \lambda_n b_n \rho_n^{-1},\ \tilde{a}_{nm} = \lambda_n a_{nm}\rho_m^{-1},\ \tilde{c}_{n+1,n} = \lambda_{n+1}c_{n+1,n}\rho_n^{-1},$$

resulting in

$$\operatorname{gcf}\left(\tilde{\mathbf{H}}\right) = \lambda_0 \operatorname{gcf}(\mathbf{H})\,\rho_0^{-1}, \qquad \tilde{K}_0^{(N)} = \lambda_0 K_0^{(N)}\rho_0^{-1}, \qquad \tilde{B}_N = \rho_N B_N \lambda_{N+1}^{-1}.$$

Theorem 7.3.1 states convergence for $\operatorname{gcf}\left(\tilde{\mathbf{H}}\right)$, and the estimate in Theorem 7.3.1 yield

$$\left|\left|\lambda_0 \left(\operatorname{gcf}(\mathbf{H}) - K_0^{(N)}\right) \rho_0^{-1}\right|\right| = \left|\left|\operatorname{gcf}\left(\tilde{\mathbf{H}}\right) - \tilde{K}_0^{(N)}\right|\right| \leq \left|\left|\tilde{B}_N^{-1}\right|\right| = \left|\left|\lambda_{N+1}B_N^{-1}\rho_0^{-1}\right|\right|.$$

□

Corollary 7.3.1. *Let $b_n \in \mathcal{R}^*$ for all $n \in \mathbb{N}_0$, let $(\lambda_n)_{n\in\mathbb{N}_0}$ be a $\mathcal{R}^*$-valued sequence, let*

$$\begin{aligned} \left|\left|\lambda_1 c_{10} b_0^{-1} \lambda_0^{-1}\right|\right| &\leq 1 \\ \left|\left|\lambda_{n+1} c_{n+1,n} b_n^{-1} \lambda_n^{-1}\right|\right| + \sum_{m=0}^{n-1} \left|\left|\lambda_m a_{mn} b_n^{-1} \lambda_n^{-1}\right|\right| &\leq 1, \qquad n \in \mathbb{N}, \end{aligned}$$

and let at least one of these inequalities be strict. Then $\operatorname{gcf}(\mathbf{H})$ *converges with* $\operatorname{gcf}(\mathbf{H}) \in \mathcal{R}^*$.

Proof. Theorem 7.3.2 yields convergence and, with $\rho_0 = \lambda_0 b_0$, we obtain

$$\left|\left|\lambda_0(\operatorname{gcf}(\mathbf{H}) - b_0) b_0^{-1} \lambda_0^{-1}\right|\right| \leq \left|\left|\lambda_1 c_{10} b_0^{-1} \lambda_0^{-1}\right|\right| \leq 1,$$

and since at least one inequality is strict, we have

$$\left|\left|\lambda_0(\operatorname{gcf}(\mathbf{H}) - b_0) b_0^{-1} \lambda_0^{-1}\right|\right| < 1,$$

yielding

$$\lambda_0 \operatorname{gcf}(\mathbf{H})\, b_0^{-1} \lambda_0^{-1} = \lambda_0(\operatorname{gcf}(\mathbf{H}) - b_0) b_0^{-1} \lambda_0^{-1} + I \in \mathcal{R}^*$$

(see Lemma A.2.1), and hence $\operatorname{gcf}(\mathbf{H}) \in \mathcal{R}^*$. □

For non-generalized continued fractions in Banach algebras, that is $a_{mn} = 0$ for $n - m > 1$, we directly obtain

Corollary 7.3.2. *Let $b_n, c_{n+1,n}, \lambda_n \in \mathcal{R}^*$, let $a_{mn} = 0$ for $n - m > 1$, let*

$$\begin{aligned} \left|\left|\lambda_2 c_{21} b_1^{-1} \lambda_1^{-1}\right|\right| &\leq 1 \\ \left|\left|\lambda_{n+1} c_{n+1,n} b_n^{-1} \lambda_n^{-1}\right|\right| + \left|\left|\lambda_{n-1} a_{n-1,n} b_n^{-1} \lambda_n^{-1}\right|\right| &\leq 1, \qquad n = 2, 3, \ldots, \end{aligned}$$

and let at least one of these inequalities be strict. Then $\operatorname{gcf}(\mathbf{H})$ *converges.*

Proof. Corollary 7.3.1 yields convergence of

$$K_1 = \lim_{N\to\infty} K_1^{(N)}$$

with $K_1 \in \mathcal{R}^*$, and thus, convergence of

$$\operatorname{gcf}(\mathbf{H}) = \lim_{N\to\infty} K_0^{(N)} = \lim_{N\to\infty} \left(b_0 + a_{01} \left(K_1^{(N)} \right)^{-1} c_{10} \right) = b_0 + a_{01} K_1^{-1} c_{10}.$$

□

Applying another equivalence transformation, we obtain

Corollary 7.3.3. *Let $b_n, c_{n+1,n}, \lambda_n \in \mathcal{R}^*$, let $a_{mn} = 0$ for $n - m > 1$, let there be numbers $q_1, q_2, \ldots \geq 1$ with*

$$\begin{aligned} 1 &\leq q_1, \\ \left|\left|\lambda_{n-1} a_{n-1,n} b_n^{-1} \lambda_n^{-1}\right|\right| \cdot \left|\left|\lambda_n c_{n,n-1} b_{n-1}^{-1} \lambda_{n-1}^{-1}\right|\right| &\leq \frac{q_n - 1}{q_{n-1} q_n}, \qquad n = 2, 3, \ldots, \end{aligned}$$

and let at least one of these inequalities be strict. Then $\operatorname{gcf}(\mathbf{H})$ *converges.*

Proof. We define

$$\tilde{\lambda}_n = \frac{1}{\prod_{m=2}^{n} q_m \left|\left|\lambda_m c_{m,m-1} b_{m-1}^{-1} \lambda_{m-1}^{-1}\right|\right|} \cdot \lambda_n$$

for $n \geq 1$, in particular $\tilde{\lambda}_1 = \lambda_1$, and we prove that the conditions of Corollary 7.3.2 are satisfied for $\tilde{\lambda}_n$. First, we have

$$\left|\left|\tilde{\lambda}_2 c_{21} b_1^{-1} \tilde{\lambda}_1^{-1}\right|\right| = \frac{\left|\left|\lambda_2 c_{21} b_1^{-1} \lambda_1^{-1}\right|\right|}{q_1 \left|\left|\lambda_2 c_{21} b_1^{-1} \lambda_1^{-1}\right|\right|} = \frac{1}{q_1} \leq 1.$$

For $n \geq 2$, we obtain

$$\begin{aligned} & \left|\left|\tilde{\lambda}_{n+1} c_{n+1,n} b_n^{-1} \tilde{\lambda}_n^{-1}\right|\right| + \left|\left|\tilde{\lambda}_{n-1} a_{n-1,n} b_n^{-1} \tilde{\lambda}_n^{-1}\right|\right| \\ = \; & \frac{\left|\left|\lambda_{n+1} c_{n+1,n} b_n^{-1} \lambda_n^{-1}\right|\right|}{q_n \left|\left|\lambda_{n+1} c_{n+1,n} b_n^{-1} \lambda_n^{-1}\right|\right|} + \left|\left|\lambda_{n-1} a_{n-1,n} b_n^{-1} \lambda_n^{-1}\right|\right| \cdot q_{n-1} \cdot \left|\left|\lambda_n c_{n,n-1} b_{n-1}^{-1} \lambda_{n-1}^{-1}\right|\right| \\ \leq \; & \frac{1}{q_n} + q_{n-1} \cdot \frac{q_n - 1}{q_{n-1} q_n} = 1. \end{aligned}$$

□

Again, we could derive an error bound similar to (3.3.8), but since the result of Theorem 3.3.5 is more general, we omit further details. Note that for $q_n = 2$, we again obtain a Worpitzky–type criterion.

7.3.3 Pringsheim-type convergence criteria for $w(w_0, \mathbf{H})$

Up to now, we considered the gcf

$$\operatorname{gcf}(\mathbf{H}) = \lim_{N \to \infty} A_N B_N^{-1}$$

itself. Since a main issue of this work is the application of gcfs to difference equations, we have to consider the convergence of $w_n(w_0, \mathbf{H})$, too. Since we already proved in a very general context that Pringsheim-type critieria guarantee that $w(w_0, \mathbf{H})$ converges and satisfies $\mathbf{H}x = 0$ up to the initial condition, we do not go too much into details, and only give one statement for demonstrating how to extend our considerations to $w(w_0, \mathbf{H})$.

Theorem 7.3.3. *Let $b_n \in \mathcal{R}^*$ for all $n \in \mathbb{N}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a $\mathcal{R}^*$-valued sequence, and let*

$$\begin{aligned} \left|\left|\lambda_2 c_{21} b_1^{-1}\lambda_1^{-1}\right|\right| &\leq 1, \\ \left|\left|\lambda_{n+1}c_{n+1,n}b_n^{-1}\lambda_n^{-1}\right|\right| + \sum_{m=1}^{n-1}\left|\left|\lambda_m a_{mn} b_n^{-1}\lambda_n^{-1}\right|\right| &\leq 1, \qquad n = 2,3,4,\ldots, \end{aligned}$$

hold. Then for all $n \in \mathbb{N}$ and $w_0 \in \mathcal{R}$, $w_n^{(N)}(w_0, \mathbf{H})$ is well-defined for sufficiently large N and converges to some $w_n(w_0, \mathbf{H})$.

Proof. By means of Corollary 7.3.1, we see that

$$K_\ell = \lim_{N\to\infty} K_\ell^{(N)} \in \mathcal{R}^*$$

for all $\ell \geq 1$. Thus, for all $\ell \geq 1$, we have

$$B_N^{(\ell)}\left(B_N^{(\ell-1)}\right)^{-1} = L_{\ell,\ell,\ell-1}^{(N)} = \left(K_\ell^{(N)}\right)^{-1} c_{\ell,\ell-1} \in \mathcal{R}^*$$

for sufficiently large N, yielding convergence of

$$B_N^{(\ell)} B_N^{-1} = \prod_1^{\ell=n}\left(K_\ell^{(N)}\right)^{-1} c_{\ell,\ell-1} \to \prod_1^{\ell=n} K_\ell^{-1} c_{\ell,\ell-1}$$

□

The results of this section are special cases of those in chapter 3. Nevertheless, since in the literature, most generalizations of continued fractions rely on the recurrence schemes for numerator and denominator, for means of comparison, it is nice to see that Pringsheim-type criteria can be proved by directly using these recursions.

7.4 Literature review

As pointed out in the introduction of this chapter, there is much literature dealing with generalizations of the recursions of numerator and denominator of continued fractions.

7.4.1 Non-generalized continud fractions defined in Banach algebras

For one-sided non-generalized continued fractions in Banach algebras, that is

$$b_0 + a_1\left(b_1 + a_2(b_2 + \ldots)^{-1}\right)^{-1},$$

the representation $K_0^{(N)} = A_N B_N^{-1}$ of the Nth approximation and the recursion $X_n = X_{n-1}b_n + X_{n-2}a_n$ has been used quite often, see [Fai71] for an early reference. Based on this representation, many approaches for proving Pringsheim–type or Worpitzky–type criteria can be found in the literature:

- Fair [Fai72] proved convergence in case of $b_n = I$ and $\sup ||a_n|| < \frac{1}{4}$, that is, a Worpitzky–type criterion. By using the concept of equivalence transformations, we easily obtain results for $b_n \in \mathcal{R}^*$. However, for $q_n = 2$ and $b_n = I$, our result in Corollary 7.3.3 states convergence even in case of $\sup ||a_n|| = \frac{1}{4}$, which is an exact generalization of Worpitzky's criterion.

- A similar result can be found in [Hay74], although it is not based on the recursion, but uses contracting mappings.

- Raisoulli and Kacha [RK00] proved convergence in case of $a_n = I$ and $\left|\left|b_{n-1}^{-1}\right|\right| \cdot \left|\left|b_n^{-1}\right|\right| \leq \frac{1}{4}$. With equivalence transformations, the authors derived a similar result in case of $a_n \in \mathcal{R}^*$. In principle, the conditions in [RK00] correspond to the Worpitzky–type criterion which can be obtained from Corollary 7.3.3 by putting $q_n = 2$, but we allow $a_n \notin \mathcal{R}^*$.

- Zhao and Zhu [ZZ03] considered the special case of $a_n \in \mathbb{C}$ and $b_n \in \mathbb{C}^{m\times m}$. Due to scalar a_n, the results do not improve those of [RK00]. Zhao and Zhu proved that 'classical' Pringsheim–type conditions are sufficient for convergence, e.g. $||B_n|| \geq 2$ in case of $a_n = 1$.

- In some way, the best results are due to Schelling [Sch96]. Schelling used the condition

$$\left|\left|b_n^{-1}\right|\right| + \left|\left|a_n b_n^{-1}\right|\right| \leq 1, \tag{7.4.1}$$

  which exactly corresponds to (7.3.6) in Theorem 7.3.1 if we put $a_{mn} = 0$ for $m < n-1$, $a_{n-1,n} = a_n$ and $c_{n+1,n} = I$. However, for his proof, he needed an additional condition, namely

$$\left|\left|a_{n-1,n} B_n^{-1}\right|\right| \cdot ||B_n|| \cdot \left|\left|b_n^{-1}\right|\right| + \left|\left|b_n^{-1}\right|\right| \leq 1, \qquad n \in \mathbb{N}. \tag{7.4.2}$$

  With an easy example, we demonstrate that (7.4.2) is restrictive: Let $\mathcal{R} = \mathbb{C}^{2\times 2}$ with the row sum norm, define $b_n = 2I$ and $a_{n-1,n} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Obviously, (7.4.1) is fulfilled. On the other hand, $B_2 = \frac{1}{2}\begin{pmatrix} 9 & 1 \\ 0 & 9 \end{pmatrix}$ yields $\left|\left|a_{12} B_2^{-1}\right|\right| \cdot ||B_2|| \cdot \left|\left|b_2^{-1}\right|\right| + \left|\left|b_n^{-1}\right|\right| = \frac{83}{81} > 1$, that is, (7.4.2) does not hold. Nevertheless, in $\mathbb{C}$, due to multiplicativity of $||\cdot|| = |\cdot|$, (7.4.2) is not restrictive. Hence, Schelling's result can be interpreted as an exact (and for $a_n \notin \mathcal{R}^*$, the first) generalization of the scalar Pringsheim-type criterion $|b_n| \geq |a_n| + 1$. Furthermore, Schelling demonstrated how to derive criteria such as Corollary 7.3.3, although – unsurprisingly – he again needed an additional (and restrictive) condition.

- Concerning Worpitzky–type criteria, the best result is due to Negoescu [Neg76]. Negoescu assumed $c_{n+1,n} \in \mathcal{R}^*$, and proved that $||a_n|| \leq \frac{1}{4}$ for $n \geq 2$ is sufficient for convergence. Furthermore, Negoescu already proved that in this case, the error bound for the $(N+1)$st approximant is $\frac{2||a_1||}{N+1}$. For tridiagonal matrices with $c_{n+1,n} = b_n = I$, these results exactly corresponds to the statements in Corollary 7.3.3 and the error bound in Corollary 3.3.1. By taking equivalence transformations, the condition $c_{n+1,n} = b_n = I$ could be replaced by $c_{n+1,n}, b_n \in \mathcal{R}^*$, we simply have to choose (λ_n) and (ρ_n) such that $\lambda_{n+1} c_{n+1,n} \rho_n^{-1} = \lambda_n b_n \rho_n^{-1} = I$. Hence, Negoescu's result completely covers Corollary 7.3.3 in case $q_n = 2$. However, as pointed out in Remark 3.3.1, it is not possible to

obtain the result for general $q_n \geq 1$ from the special case $q_n = 2$ by means of equivalence transformations. Hence, the results in Corollary 7.3.3 and Theorem 7.3.2 are slightly stronger than those in [Neg76]. Nevertheless, Negoescu's results were the first generalizations of Pringsheim-type criteria to Banach algebras.

- For non-generalized continued fractions with $c_{n+1,n} \in \mathcal{R}^*$, the results of this chapter have been published in [Bau13].

7.4.2 n-fractions defined in $\mathbb{C}$

Up to now, we discussed literature dealing with (one-sided) **non-generalized** continued fraction based on coefficients $a_n, b_n \in \mathcal{R}$ with some Banach algebra $\mathcal{R}$. Based on generalizations of the recurrence scheme for numerator and denominator, we find many approaches for generalized continued fractions in the literature, most of them with domain $\mathbb{C}$.

The first generalization of the recurrence for A_n and B_n is due to Jacobi. In [Jac68], he gave the fundaments for the Jacobi–Perron algorithm, which was extended by the work of Perron [Per07; Per09]. With our notation, Perron defined a *Jacobi–chain of order n* by

$$\begin{aligned} B_m^{(\ell)} &= \delta_{m\ell}, \qquad m, \ell = 0, \ldots, n, \\ B_m^{(\ell)} &= B_{m-1}^{(\ell)} b_m + \sum_{k=m-n}^{m-1} B_{k-1}^{(\ell)} a_{km}, \qquad m > n, \ell = 0, \ldots, n, \\ K &= \left(\lim_{m\to\infty} a_{0n} \frac{B_m^{(\ell)}}{B_m^{(0)}} \right)_{\ell=1}^{n}, \end{aligned}$$

if the limits exist. For the coefficients b_m, a_{km}, he allowed arbitrary complex numbers up to the condition $a_{m-n,m} \neq 0$ for all $m > n$. Furthermore, he gave a continued-fraction interpretation for the entries of K, and he proved a Pringsheim-type criterion for convergence: If

$$\sum_{k=m-n}^{m-1} |a_{km}| \leq \vartheta(|b_m| - 1)$$

for all $m \geq n + 2$ and some real constant $\vartheta < 1$, the Jacobi-chain converges. We may interprete K as a special case of the vector $(L_{1,1,0}, \ldots, L_{n,1,0})$, and our Pringsheim–type criteria guarantee convergence even in case of $\vartheta = 1$. For an extensive study of the Jacob–Perron algorithm, we refer to [Ber71].

A quite similar generalization of continued fractions is due to de Bruin [Bru74; Bru78]. Up to notation, his approach is as follows: Let the (complex-valued) sequences $\left(A_N^{(-n)}\right)_{N\geq -n}$, $\ldots, \left(A_N^{(-1)}\right)_{N\geq -n}$, $(B_N)_{N\geq -n}$ satisfy the recurrence relation

$$X_N = X_{N-1} b_N + \sum_{m=N-n}^{N-1} X_{m-1} a_{mN}, \qquad N \in \mathbb{N},$$

subject to the initial conditions

$$\begin{pmatrix} A_{-n}^{(-n)} & A_{-n+1}^{(-n)} & \dots & A_{-1}^{(-n)} & A_0^{(-n)} \\ A_{-n}^{(-n+1)} & A_{-n+1}^{(-n+1)} & \dots & A_{-1}^{(-n+1)} & A_0^{(-n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{-n}^{(-1)} & A_{-n+1}^{(-1)} & \dots & A_{-1}^{(-1)} & A_0^{(-1)} \\ B_{-n} & B_{-n+1} & \dots & B_{-1} & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & a_{-n+1,0} \\ 0 & 1 & \dots & 0 & a_{-n+2,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then define the *n-fraction*

$$K = \lim_{N\to\infty} \left(\frac{A_N^{(-n)}}{B_N}, \dots, \frac{A_N^{(-1)}}{B_N} \right)$$

We may generalize de Bruin's approach as follows by defining vector-valued gcfs, and in particular, n-fractions, in a Banach algebra $\mathcal{R}$.

Definition 7.4.1. Let $n \in \mathbb{N}$, $b_N = 0$ and $a_{mN} = 0$ for $N < 0$, and let $c_{N+1,N} \in \mathcal{R}^*$ for all $N \geq -n$. Define

$$A_{-\ell}^{(-\ell)} = I,\ \ell = 1, \dots, n, \qquad A_N^{(-\ell)} c_{N+1,N} = A_{N-1}^{(-\ell)} b_N + \sum_{m=-\ell+1}^{N-1} A_{m-1}^{(\ell)} a_{mN},\ N > -\ell$$

$$B_0 c_{10} = I, \qquad B_N c_{N+1,N} = B_{N-1} b_N + \sum_{m=1}^{N-1} B_{N-1} a_{mN},\ N \in \mathbb{N}.$$

If $B_N \in \mathcal{R}^*$ for almost all $N \in \mathbb{N}$ and if

$$K = \lim_{N\to\infty} \left(A_N^{(-C)} B_N^{-1}, \dots, A_N^{(-1)} B_N^{-1} \right)$$

exists, K is said to be a (n-dimensional) vector-valued gcf. In case of $a_{mN} = 0$ for $N - m > n$, K is said to be an n-fraction.

For $c_{N+1,N} = I$ and $\mathcal{R} = \mathbb{C}$, our definition of n-fractions and that due to de Bruin coincide. Since the recursions for B_N and $A_N = A_N^{(-1)}$ have not changed in comparison to previous considerations, for the last entry of the vector-valued gcf, we already have a Pringsheim–type convergence criterion. Now consider $\ell > 1$. Define $\tilde{A}_N$ by $\tilde{A}_{-1} = I$ and $\tilde{A}_N c_{N+1,N} = \tilde{A}_{N-1}\tilde{b}_N + \sum_{m=0}^{N-1} \tilde{A}_{m-1}\tilde{a}_{mN}$ where $\tilde{b}_0 = a_{-\ell+1,0}$, $\tilde{a}_{0N} = a_{-\ell+1,N}$ and $\tilde{b}_N = b_N$, $\tilde{a}_{mN} = a_{mN}$ for $m \geq 1$. Then due to $A_N^{(-\ell)} = 0$ for $N = -\ell+1, \dots, -1$, we have $\tilde{A}_N = A_N^{(-\ell)}$ for all $N \in \mathbb{N}_0$, and thus, we have a representation

$$A_N^{(-\ell)} B_N^{-1} = \tilde{A}_N B_N^{-1}.$$

Note that the values of b_0 and a_{0N} do not influence B_N. Hence, we may apply our criteria to $A_N^{(-\ell)} B_N^{-1}$ for $\ell \geq 2$. These considerations result in

Theorem 7.4.1. *Let $b_N \in \mathcal{R}^*$ for all $N \in \mathbb{N}$ and let*

$$\left|\left|b_N^{-1}\right|\right| + \sum_{m=1}^{N-1} \left|\left|a_{mN} b_N^{-1}\right|\right| + \max_{\ell=1,\dots,n} \left|\left|a_{-\ell+1,N} b_N^{-1}\right|\right| \leq 1$$

hold for all $N \in \mathbb{N}$. Then the vector-valued gcf

$$K = \lim_{N\to\infty} \left(A_N^{(-C)} B_N^{-1}, \dots, A_N^{(-1)} B_N^{-1}\right)$$

converges. In particular, for n-fractions, a sufficient criterion for convergence is given by $b_N \in \mathcal{R}^*$ with

$$\left|\left|b_N^{-1}\right|\right| + \sum_{m=1}^{N-1} \left|\left|a_{mN} b_N^{-1}\right|\right| + \max_{\ell=1,\dots,n-n+1} \left|\left|a_{-\ell+1,N} b_N^{-1}\right|\right| \leq 1, \qquad N = 1, \dots, n-1,$$

$$\left|\left|b_N^{-1}\right|\right| + \sum_{m=N-n}^{N-1} \left|\left|a_{mN} b_N^{-1}\right|\right| \leq 1, \qquad N \geq n.$$

This result is an exact generalization of Levrie's results for $\mathbb{C}$-valued n-fractions, [Lev86; Lev89]. Vector-valued gcfs or n-fractions are strongly connected to solutions of sum equations or difference equations. In $\mathbb{C}$, this approach is quite similar to our 'inverse approach' $w_n^{(N)}(w_0, \mathbf{H})$. In Banach algebras, we prefer using $w^{(N)}$ in order to save some unnecessary inversions. To the knowledge of the author, n-fractions in Banach algebras have not been considered so far.

7.4.3 Volterra difference equations for A_n and B_n

In principle, n-fractions are generated by upper Hessenberg matrices with $a_{mN} = 0$ for $N - m > n$. Without this assumption, the recursion

$$X_N c_{N+1,N} = X_{N-1} b_N + \sum_{m=0}^{n} X_{m-1} a_{mN}$$

for A_N and B_N is a Volterra difference equation. In $\mathbb{C}$, Perron [Per20] considered these recursions and referred to $\lim_{N\to\infty} \frac{A_N}{B_N}$ as 'infinite Jacobi chain'. To the knowledge of the author, neither a Pringsheim–type criterion nor a direct generalization to Banach algebras has been published so far. Nevertheless, Perron stated relationships between gcfs and sum equations, and hence, many of our results (in this chapter, in chapter 4 and in chapter 8) can be interpreted as generalizations of Perron's results.

7.4.4 Matrix continued fractions according to Levrie and Bultheel

Another kind of generalized continued fraction in Banach algebras is due to Levrie and Bultheel [LB96]. Up to notation, their idea is as follows: The recurrence relation $X_n =$

$X_{n-1}b_n + X_{n-2}a_n$ for numerators A_n and denominators B_n of non-generalized continued fractions can be written as

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{pmatrix} \cdot \begin{pmatrix} b_n & I \\ a_n & 0 \end{pmatrix},$$

subject to $\begin{pmatrix} A_0 & A_{-1} \\ B_0 & B_{-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, a kind of generalization of continued fractions can be obtained from defining

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \prod_{k=0}^{n} \theta_k \cdot \begin{pmatrix} I \\ 0 \end{pmatrix}$$

where $\theta_k = \begin{pmatrix} b_k & c_k \\ a_k & d_k \end{pmatrix}$. Levrie and Bultheel referred to $\lim_{n\to\infty} A_n B_n^{-1}$ as 'matrix continued fraction'. They assumed b_k, c_k, a_k, d_k to be matrices with dimensions independent of k, θ_k being a quadratic matrix. Here, we assume that all coefficients are elements of some Banach algebra $\mathcal{R}$. In order to guard against misunderstandings (in the literature, the term 'matrix continued fractions' is also used for continued fractions with matrix-valued coefficients, see [Pfl66; RK00; SI99; ZZ03]), we will not use this term in this context. Instead, we will refer to Levrie and Bultheel's construction as *LB-fractions*.

Actually, LB-fractions are quite strongly related to gcfs: Writing $\prod_{k=0}^{n} \theta_k = \begin{pmatrix} A_n & C_n \\ B_n & D_n \end{pmatrix}$ yields

$$\begin{aligned} A_n &= A_{n-1}b_n + C_{n-1}a_n \\ &= A_{n-1}b_n + (A_{n-2}c_{n-1} + C_{n-2}d_{n-1})\, a_n = \dots \\ &= A_{n-1}b_n + A_{n-2}c_{n-1}a_n + A_{n-3}c_{n-2}d_{n-1}a_n + A_{n-4}c_{n-3}d_{n-2}d_{n-1}a_n \\ &\quad + \dots + A_0 c_1 d_2 \dots d_{n-1}a_n + C_0 d_1 \dots d_{n-1}a_n \end{aligned}$$

and

$$\begin{aligned} B_n &= B_{n-1}b_n + B_{n-2}c_{n-1}a_n + B_{n-3}c_{n-2}d_{n-1}a_n + B_{n-4}c_{n-3}d_{n-2}d_{n-1}a_n \\ &\quad + \dots + B_0 c_1 d_2 \dots d_{n-1}a_n + D_0 d_1 \dots d_{n-1}a_n. \end{aligned}$$

Hence, with

$$a_{mn} = c_m \prod_{k=m+1}^{n-1} d_k a_n,$$

(A_n) and (B_n) satisfy the usual recurrence scheme at least in case of $C_0 = A_{-1}$ and $D_0 = 0$. Therefore, if $C_0 = c_0 = I$, $B_0 = a_0 = I$ and $D_0 = d_0 = 0$, we can directly apply Theorem 7.3.1:

Theorem 7.4.2. *Let $a_0 = c_0 = I$ and $d_0 = 0$, let $b_n \in \mathcal{R}^*$ with*

$$\left|\left|b_n^{-1}\right|\right| + \sum_{m=0}^{n-1} \left|\left| c_m \prod_{k=m+1}^{n-1} d_k a_n b_n^{-1} \right|\right| \leq 1$$

for all $n \in \mathbb{N}$. *Then*

$$\lim_{N\to\infty} A_N B_N^{-1}$$

exists, where

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \prod_{k=0}^{n} \theta_k \cdot \begin{pmatrix} I \\ 0 \end{pmatrix} \qquad \textit{and} \qquad \theta_k = \begin{pmatrix} b_k & c_k \\ a_k & d_k \end{pmatrix} \theta_k = \begin{pmatrix} b_k & c_k \\ a_k & d_k \end{pmatrix}.$$

Naturally, applying equivalence transformations would yield more general results. Since we will not focus on LB–fractions, we omit these considerations. Levrie and Bultheel themselves proved a Pincherle–type criterion for convergence, a similar approach can be found in [Ahl96]. We will derive criteria of this type in chapter 8.

7.4.5 Vector–valued recurrence schemes

In $\mathbb{C}$, even more generalized recurrence schemes have been considered. Hanschke [Han98] considered the equation

$$x_{n+1} = x_n A_n$$

for $x_n \in \mathbb{C}^r$ and $A_n^{r\times r}$, and found criteria for convergence of

$$\frac{\langle x_n^{(i)}, \sigma_n \rangle}{\langle x_n^{(k)}, \sigma_n \rangle},$$

where (σ_n) is a given sequence of r-dimensional vectors and $x^{(1)}, \ldots, x^{(r)}$ are linearly independent solutions of the difference equation. As for LB-fractions, we could derive infinite recursions for the entries of x_n, and hence, obtain gcfs.

Chapter 8

Dominance and Subdominance

In section 4.8, we pointed out that gcfs will characterize minimal solutions of systems of equations under some assumptions. As we have seen in chapter 5, this is in particular true for applications in the context of Markov chains. In section 1.1 in the introduction and in section 6.3, we have seen examples in which these solutions were subdominant.

Therefore, the question arises whether or not gcfs characterize a subdominant solution of the sum equation

$$c_{n,n-1}x_{n-1} = b_n x_n + \sum_{m=n+1}^{\infty} a_{nm}x_m, \qquad n \in \mathbb{N},$$

which is obtained from $\mathbf{H}x = 0$ for upper Hessenberg matrices. As we will demonstrate in section 8.1, for upper Hessenberg matrices, the gcf-based method is quite strongly related to Miller's backward computation. It is well-known that for second-order difference equations ($a_{nm} = 0$ for $m \geq n+2$) in $\mathcal{R} = \mathbb{C}$, backward computation is guaranteed to compute subdominant solutions, whereas there are counterexamples for difference equations of higher order. Hence, due to the strong relationship to gcf-based solutions, the answer to the above question is negative. For sake of completeness, we will reflect an counterexample from the literature in section 8.1.

Instead, for n-fractions in the sense of de Bruin (see section 7.4), literature ([Cru79; Cru82]) suggests to characterize convergence of gcfs in terms of the existence of a dominant solution of the adjoint system which is basically $\psi\mathbf{H} = 0$ (in section 8.2, we will give more details on the concept of adjoint systems).

In section 8.3, we will generalize the results from [Cru79; Cru82] to the context of gcfs defined in the sense of chapter 7. It will turn out that the asymptotic behaviour of the solutions of the adjoint system not only characterizes whether or not a gcf converges, but also determines its value in case of convergence.

8.1 Miller's algorithm

In this section, we want to demonstrate that the construction of gcf-based solutions for $\mathbf{H}x = 0$ is strongly related to Miller's algorithm. Originally, Miller was interested in a stable computation method for Airy integrals and for Bessel functions. We refer to [Gau67; Gau72; Olv85;

Wim84] for details and comments on the evolution of the algorithm. Modified Bessel-functions of the first kind, $I_n(z)$, satisfy the second-order difference equation

$$x_{n-1}(z) = \frac{2n}{z}x_n(z) + x_{n+1}(z), \qquad n \in \mathbb{N},\ z > 0. \tag{8.1.1}$$

The second solution is given by $(-1)^n K_n(z)$, where $K_n(z)$ are the modified Bessel-functions of the second kind. For fixed z and $n \to \infty$, we have $\frac{I_n(z)}{K_n(z)} \to 0$ (see [Wim84]), and therefore, even if we know the exact values of $I_0(z)$ and $I_1(z)$, $I_n(z)$ cannot be computed from (8.1.1) by forward computation, since this procedure becomes unstable (similar to the Markov Chain example 6.3.1). Miller suggested the following algorithm:

- Choose N large and put $\tilde{w}_{N+1} = 0$, $\tilde{w}_N = 1$.
- For $n = N, N-1, \ldots, 1$ compute $\tilde{w}_{n-1}$ by (8.1.1).
- For $n = 0, \ldots, N$ set $w_n = \frac{I_0(z)\tilde{w}_n}{\tilde{w}_0}$.

The last step is a kind of normalization, guaranteeing $w_0 = I_0(z)$. The backward computation, which is the key idea of Miller's algorithm, yields that (8.1.1) is fulfilled for $n = 1, \ldots, N$. If w_n converges for $N \to \infty$, it is clear that the limit satisfies (8.1.1) for all $n \in \mathbb{N}$. Nevertheless, at a first glance, it is not clear that w_n converges to $I_n(z)$ for $N \to \infty$, since for all $c \in \mathbb{C}$, the sequence (x_n) with

$$x_n = cI_n(z) + (1-c)\frac{I_0(z)}{K_0(z)}(-1)^n K_n(z)$$

satisfies (8.1.1) and $x_0 = I_0(z)$. At this point, the subdominance of the solution $I_n(z)$ becomes important. For second-order linear difference equations in $\mathbb{C}$, it can be proved that Miller's algorithm always converges to the subdominant solution if there is one, see [Gau72].

We give the key idea for a proof here, using a slightly more general setting than found in the literature. In a Banach algebra $\mathcal{R}$, we consider the second-order linear difference equation

$$c_{n,n-1}x_{n-1} = b_n x_n + a_{n,n+1}x_{n+1}, \qquad n \in \mathbb{N}, \tag{8.1.2}$$

where $c_{n,n-1} \in \mathcal{R}^*$ for all $n \in \mathbb{N}$, and let $x^{(1)} = \left(x_n^{(1)}\right)$ and $x^{(2)} = \left(x_n^{(2)}\right)$ be two linearly independent solutions with $x_0^{(1)} = x_0^{(2)} = I$ and $x_n^{(2)} \in \mathcal{R}^*$ for almost all $n \in \mathbb{N}$.

Now, we apply Miller's algorithm (which is possible due to the invertibility of $c_{n,n-1}$), where the normalization shall guarantee $w_0 = I$. By $\tilde{w}_n^{(N)}, w_n^{(N)}$ we denote the results of Miller's algorithm, where the normalization step is given by $w_n^{(N)} = \tilde{w}_n^{(N)}\left(\tilde{w}_0^{(N)}\right)^{-1}$, assuming $\tilde{w}_0^{(N)} \in \mathcal{R}^*$ for almost all $N \in \mathbb{N}$. Finally, in case of existence, we write $w_n = \lim w_n^{(N)}$. Since we have $\tilde{w}_{N+1}^{(N)} = 0$ and $\tilde{w}_N^{(N)} = I$, and due to our assumptions on $x^{(1)}$ and $x^{(2)}$, we can write

$$\tilde{w}_n^{(N)} = \left(x_n^{(1)} - x_n^{(2)}\left(x_{N+1}^{(2)}\right)^{-1}x_{N+1}^{(1)}\right)\left(x_N^{(1)} - x_N^{(2)}\left(x_{N+1}^{(2)}\right)^{-1}x_{N+1}^{(1)}\right)^{-1},$$

if additionally this inverse exists. The normalization yields

$$w_n^{(N)} = \left(x_n^{(1)} - x_n^{(2)}\left(x_{N+1}^{(2)}\right)^{-1}x_{N+1}^{(1)}\right)\left(I - \left(x_{N+1}^{(2)}\right)^{-1}x_{N+1}^{(1)}\right)^{-1},$$

if additionally the latter inverse exists. Now assume that $x^{(1)}$ is dominated by $x^{(2)}$ in the sense of

$$\lim_{N\to\infty} \left(x_N^{(2)}\right)^{-1} x_N^{(1)} = 0,$$

then it is clear that $w_n^{(N)}$ converges to $x_n^{(1)}$. Due to many invertibility assumptions, this is no honest proof, but it demonstrates why Miller's algorithm was developed for characterizing subdominant solutions.

The first problem is that for $\mathcal{R} \neq \mathbb{C}$, the assumption $c_{n,n-1} \in \mathcal{R}^*$ becomes more restrictive. In the sketch of the proof of Miller's algorithm, some further inverses are involved, and we do not have any guarantee for their existence neither. In this context, our setting of truncated systems of equations and continued-fraction-based methods becomes useful. Miller's algorithm is based on solving the truncated difference equation

$$c_{n,n-1}x_{n-1} = b_n x_n + a_{n,n+1}x_{n+1}, \qquad n = 1, \ldots, N,$$

subject to the boundary conditions $x_{N+1} = 0$ and $x_N = I$. In applications, usually the value x_0 is given, and this changes the second boundary condition: We do not know $w_N^{(N)}$, we have to choose it in such a way that $w_0^{(N)}$ will coincide with the given value x_0. For this purpose, we use the truncated system, and derive

$$w_N^{(N)} = b_N^{-1}c_{N,N-1}w_{N-1}^{(N)},\ w_{N-1}^{(N)} = \left(b_{N-1} + a_{N-1,N}b_N^{-1}c_{N,N-1}\right)^{-1} w_{N-2}^{(N)},$$

and by induction, we see that $w_n^{(N)} = L_{n,n,n-1}^{(N)} w_{n-1}^{(N)}$. Therefore,

$$w_N^{(N)} = L_{N,N,N-1}^{(N)} \cdot \ldots \cdot L_{1,1,0}^{(N)} x_0 = L_{n,1,0}^{(N)} x_0 = w_n^{(N)}(x_0, \mathbf{H})$$

is the appropriate choice for guaranteeing $w_0^{(N)} = x_0$. From this equation, we directly see that Miller's algorithm is similar to the gcf-based method. In some way, Miller's algorithm is a pure backward computation, whereas the gcf-based method is a kind of backward-forward computation. In general, the methods are not equivalent, since there are distinct values to invert. A straight-forward generalization of the ideas in the last lines yields

Theorem 8.1.1. *Let* $\mathbf{H}$ *be an upper Hessenberg-matrix with* $c_{n+1,n} \in \mathcal{R}^*$ *and let the approximation* $w^{(N)} = w^{(N)}(w_0, \mathbf{H})$ *be well-defined. Then* $w^{(N)}$ *can be computed by backward computation (Miller's algorithm), that is*

- *Put* $\tilde{w}_{N+1}^{(N)} = \tilde{w}_{N+2}^{(N)} = \ldots = 0$
- *For* $n = N, N-1, \ldots, 1$ *use the sum equation* (8.2.1) *for computing* $\tilde{w}_{n-1}^{(N)}$.
- *Put* $w_n^{(N)} = \tilde{w}_n^{(N)} \left(\tilde{w}_0^{(N)}\right)^{-1} w_0$.

We have already seen that gcf-based solutions for $\mathbf{H}x = 0$ are available in a much more general context, whereas the backward computation of Miller's algorithm is restricted to upper Hessenberg matrices with $c_{n+1,n} \in \mathcal{R}^*$.

Whereas for second-order difference equations Miller's algorithm or the gcf-based method is strongly connected to subdominant solutions, this is not true in general. In Example 4.2.1 we already discussed a sum equation for which $w_n^{(N)}$ converges to numbers w_n, but the sequence (w_n) is no solution of the sum equation. Precisely, there was no solution at all in Example 4.2.1. If in $\mathbb{C}$, $a_{mn} = 0$ for $n - m > C$, we have a difference equation which has C linearly independent solutions. Furthermore, for difference equations it is clear that – as long as the limits $w_n = \lim w_n^{(N)}$ exist – (w_n) will always satisfy the difference equation. Nevertheless, for $C \geq 3$ it is possible to construct examples, where the gcf-based solution (or the solution generated by Miller's algorithm) exists, solves the difference equation, but dominates another solution.

One such example is given by Zahar [Zah77]: Using our notation, he considers the third-order difference equation

$$c_{n,n-1}x_n = b_n x_n + a_{n,n+1}x_{n+1} + a_{n,n+2}x_{n+2}, \qquad n \geq 2 \tag{8.1.3}$$

where

n	even	odd
$c_{n,n-1}$	2	$1+\sqrt{\frac{n+2}{n}}$
b_n	$-3+\sqrt{\frac{n+1}{n-1}}$	-2
$a_{n,n+1}$	$-2\sqrt{\frac{n+1}{n-1}}$	$1-3\sqrt{\frac{n+2}{n}}$
$a_{n,n+2}$	$1+\sqrt{\frac{n+1}{n-1}}$	$2\sqrt{\frac{n+2}{n}}$

Three linearly independent solutions $x^{(j)} = \left(x_n^{(j)}\right)_{n=1}^{\infty}$, $j = 1, 2, 3$ are given by

$$x_n^{(1)} = 1, \qquad x_n^{(2)} = n, \qquad x_n^{(3)} = \begin{cases} 0, & n \text{ even}, \\ \frac{1}{\sqrt{n}}, & n \text{ odd}. \end{cases}$$

Clearly, $x^{(3)}$ is dominated by all other solutions, but Miller's backward computation, and thus, the (due to $\mathcal{R} = \mathbb{C}$) equivalent gcf-method converges to $x^{(1)}$ (see [Zah77] for more details).

This example demonstrates that even for difference equations, Miller's backward computation or the gcf-method will not characterize subdominant solutions in any case. In addition to the example, we give some theoretical considerations: In $\mathbb{C}$, consider a general third-order difference equation

$$c_{n,n-1}x_n = b_n x_n + a_{n,n+1}x_{n+1} + a_{n,n+2}x_{n+2}, \qquad n \in \mathbb{N} \tag{8.1.4}$$

with linearly independent solutions $x^{(1)}, x^{(2)}, x^{(3)}$. $w_n^{(N)}$ solves the difference equation for $n = 1, \ldots, N$, subject to $w_0^{(N)} = w_0$, $w_{N+1}^{(N)} = w_{N+2}^{(N)} = 0$. Using Cramer's rule it is quite easy to obtain

$$w_n^{(N)} = w_0 \cdot \frac{\left(x_{N+1}^{(2)}x_{N+2}^{(3)} - x_{N+1}^{(3)}x_{N+2}^{(2)}\right)x_n^{(1)} + \left(x_{N+1}^{(3)}x_{N+2}^{(1)} - x_{N+1}^{(1)}x_{N+2}^{(3)}\right)x_n^{(2)} + \left(x_{N+1}^{(1)}x_{N+2}^{(2)} - x_{N+1}^{(2)}x_{N+2}^{(1)}\right)x_n^{(3)}}{\left(x_{N+1}^{(2)}x_{N+2}^{(3)} - x_{N+1}^{(3)}x_{N+2}^{(2)}\right)x_0^{(1)} + \left(x_{N+1}^{(3)}x_{N+2}^{(1)} - x_{N+1}^{(1)}x_{N+2}^{(3)}\right)x_0^{(2)} + \left(x_{N+1}^{(1)}x_{N+2}^{(2)} - x_{N+1}^{(2)}x_{N+2}^{(1)}\right)x_0^{(3)}}.$$

Therefore, an appropriate assumption yielding convergence of $w_n^{(N)}$ to $\frac{w_0}{x_0^{(1)}} x_n^{(1)}$ is

$$\frac{x_{N+1}^{(3)} x_{N+2}^{(1)} - x_{N+1}^{(1)} x_{N+2}^{(3)}}{x_{N+1}^{(2)} x_{N+2}^{(3)} - x_{N+1}^{(3)} x_{N+2}^{(2)}} \to 0 \text{ and } \frac{x_{N+1}^{(1)} x_{N+2}^{(2)} - x_{N+1}^{(2)} x_{N+2}^{(1)}}{x_{N+1}^{(2)} x_{N+2}^{(3)} - x_{N+1}^{(3)} x_{N+2}^{(2)}} \to 0,$$

and this is not equivalent to subdominance of $x^{(1)}$. In Zahar's example, this assumption is true, since the first fraction is $\sim \frac{N^{-\frac{1}{2}}}{N \cdot N^{-\frac{1}{2}}}$ and the second fraction is $\sim \frac{1}{N \cdot N^{-\frac{1}{2}}}$. For difference equations in $\mathbb{C}$, the above condition can be generalized in terms of determinants, see [Han98, Theorem 3.6].

Nevertheless, we have already seen that under some conditions (positivity, convergent S-series) the gcf-based method characterizes a *minimal* solution (see section 4.8), which cannot dominate other solutions. Furthermore, we will see that for periodic coefficients (chapter 9), the gcf-based solution is strongly connected to minimal roots of an meromorphic function.

Zahar [Zah77] states that Miller's algorithm converges if and only if the adjoint system has a dominant solution. In terms of continued fractions, the adjoint system can be interpreted as the recurrence scheme for numerator(s) and denominator. A result due to van der Cruyssen [Cru79] states that a n-fraction converges if and only if this recurrence scheme has a dominant solution. Such criteria are known as Pincherle–type criteria. Together, van der Cruyssen's result and Zahar's characterization of convergence of Miller's algorithm yield that Miller's algorithm converges if and only if the corresponding n-fraction converges. From Theorem 8.1.1, this is already clear in the more general context of sum equations in $\mathbb{C}$. However, we want to develop similar relationships for gcfs, and hence, we will discuss some general properties of adjoint systems, and then, we will prove Pincherle–type convergence criteria for gcfs.

8.2 Adjoint systems

As pointed out above, for upper Hessenberg-matrices, $\mathbf{H}x = 0$ yields a *sum equation* and a boundary condition, that is

$$c_{n,n-1} x_n = b_n x_n + \sum_{m=n+1}^{\infty} a_{mn} x_n, \qquad n = 1, 2, 3, \ldots, \tag{8.2.1}$$

$$0 = b_0 x_0 + \sum_{n=1}^{\infty} a_{0n} x_n, \tag{8.2.2}$$

and we know that $w(w_0, \mathbf{H})$ is a candidate for solving (8.2.1). By an appropriate choice of w_0, that is $w_0 \text{gcf}\,(\mathbf{H}) = 0$, $w(w_0, \mathbf{H})$ might solve the boundary condition, too.

On the other hand, $\psi\mathbf{H} = 0$ can be rewritten as a system of *Volterra difference equations*, that is

$$(V_n) \qquad \psi_{n+1} c_{n+1,n} = \psi_n b_n + \sum_{m=0}^{n-1} \psi_m a_{mn}, \qquad n = 0, 1, 2, \ldots$$

which obviously coincides with the recursion (7.1.1) for $\left(B_n^{(\ell)}\right)$, that is, for all $\ell \geq -1$, $\left(B_n^{(\ell)}\right)_{n \in \mathbb{N}_0}$ solves (V_n) for $n \geq \ell + 1$. In section 8.3, we will characterize convergence of gcfs

in terms of the existence of a dominant solution for $(V_n)_{n\geq 1}$. Convergence criteria of this type are referred to as *Pincherle-type criteria*.

In this context, we will refer to (V_n) as the *adjoint system* for (8.2.1). In order to guard against misunderstandings, we compare our usage of adjoint systems with a quite popular definition of adjoint sequences (for $\mathcal{R} = \mathbb{C}$, see [Zah77] for example).

Definition 8.2.1. Let $\ell \in \mathbb{N}_0$, let $(U_n)_{n>\ell}$ be a sequence of matrix $U_n \in \mathcal{R}^{d_n \times d_{n+1}}$ with $d_n \in \mathbb{N}$ for all $n \geq \ell + 1$. Then the recurrences

$$\xi_{n-1} = U_n \xi_n,\ n \geq \ell + 1 \qquad \text{and} \qquad \eta_n = \eta_{n-1} U_n,\ n \geq \ell + 1$$

for sequences $(\xi_n)_{n\geq \ell}$, $(\eta_n)_{n\geq \ell}$ of $\mathcal{R}$-valued vectors of appropriate size are said to be *adjoint*.

In order to demonstrate that the sum equation (8.2.1) and the Volterra-difference equations (V_n) are adjoint in the sense of Definition 8.2.1, we have to transform these relations into vector-valued first-order relations. Here, without restriction, we assume $c_{n+1,n} = I$ for all $n \in \mathbb{N}_0$, and define

$$\eta_n = (\psi_{n+1}, \psi_n, \ldots, \psi_0).$$

If (ψ_n) solves (V_n) for $n \geq \ell + 1$, we have

$$\eta_n = \eta_{n-1} \begin{pmatrix} b_n & 1 & & & \\ a_{n-1,n} & & 1 & & \\ a_{n-2,n} & & & 1 & \\ \vdots & & & & \ddots \\ a_{0n} & & & & & 1 \end{pmatrix} =: \eta_{n-1} U_n$$

for $n \geq \ell+1$. In principle, this is the standard way of transforming higher-order relations into first-order relations. If (x_n) is a solution of the sum equation (8.2.1), the standard method $\xi_n = (x_n, x_{n+1}, \ldots)^T$ does not work (that is, we do not obtain $\xi_{n-1} = U_n \xi_n$). Instead, we put

$$\xi_n = \begin{pmatrix} x_n \\ \sum\limits_{m=n+1}^{\infty} a_{nm} x_m \\ \sum\limits_{m=n+1}^{\infty} a_{n-1,m} x_m \\ \vdots \end{pmatrix}.$$

Then, we have indeed $\xi_{n-1} = U_n \xi_n$, and thus, adjoint recurrences. We conclude this section with a simple result concerning adjoint sequences.

Lemma 8.2.1. *Let (U_n), (ξ_n) and (η_n) as in Definition* 8.2.1. *Then*

$$\eta_\ell \xi_\ell = \eta_n \xi_n, \qquad n \geq \ell. \tag{8.2.3}$$

Proof. By definition, we have

$$\xi_\ell = U_{\ell+1} U_{\ell+2} \ldots U_n \xi_n \text{ and } \eta_n = \eta_\ell U_{\ell+1} U_{\ell+2} \ldots U_n.$$

Multiplication directly yields

$$\eta_\ell \xi_\ell = \eta_\ell U_{\ell+1} U_{\ell+2} \ldots U_n \xi_n = \eta_n \xi_n.$$

□

8.3 Pincherle-type convergence criteria

Briefly summarized, Pincherle-type criteria characterize convergence of a continued fraction or some generalization in terms of the existence of some dominant solution of the adjoint system, that is the recurrence scheme for numerator(s) and denominator.

Definition 8.3.1. Consider a difference equation of order $C+1$ in $\mathbb{C}$.

- A solution (Y_n) is said to be a *subdominant* solution if $\frac{Y_n}{Z_n} \to 0$ for any solution (Z_n) with $(Y_n), (Z_n)$ being linearly independent.
- Let $\left\{(Z_n), \left(Y_n^{(1)}\right), \ldots, \left(Y_n^{(C)}\right)\right\}$ be a fundamental system of solutions of the difference equation. (Z_n) is said to be a *dominant* solution if $\frac{Y_n}{Z_n} \to 0$ for any solution $(Y_n) \in$ *span* $\left\{\left(Y_n^{(1)}\right), \ldots, \left(Y_n^{(C)}\right)\right\}$.

For ordinary continued fraction, a well-known result states that the continued fraction converges if and only if there is some subdominant solution for the recursion for numerator and denominator, see [Gau67]. Due to the fact that this recursion is a second-order difference equation, it is clear that there is also a dominant solution in this case. For n-fractions, difference equations of order $n+1$ have to be considered, and it turned out that the n-fraction converges if and only if there is some dominant solution, see [Cru79]. Therefore, it seems reasonable to conjecture that for any generalization of continued fractions, there is some characterization of convergence in terms of the existence of a dominant solution of the adjoint system (which defines the recurrence schemes for numerator and denominator).

In the next results, we will state Pincherle-type convergence criteria for gcfs, using ideas similar to [Gau67], [Cru79] or [LB96]. In comparison to these proofs, we have to be slightly more careful when using inverses.

Theorem 8.3.1. *Let* $\mathbf{H}$ *be an upper Hessenberg matrix with* $c_{n+1,n} \in \mathcal{R}^*$ *for all* $n \in \mathbb{N}_0$. *The continued fraction* $\operatorname{gcf}(\mathbf{H})$ *converges if and only if there are solutions* $(Y_n), (Z_n)$ *for*

$$X_n c_{n+1,n} = X_{n-1} b_n + \sum_{m=0}^{n-1} X_{m-1} a_{mn}, \qquad n = 1, 2, \ldots \tag{8.3.1}$$

with

- $Y_{-1}, Z_0 - Z_{-1} Y_{-1}^{-1} Y_0 \in \mathcal{R}^*$,
- $Z_n \in \mathcal{R}^*$ *for almost all* n,
- $\lim_{n\to\infty} Y_n Z_n^{-1} = 0$.

In this case, we have

$$\operatorname{gcf}(\mathbf{H}) = b_0 - Y_{-1}^{-1} Y_0 c_{10}.$$

Proof. First, we assume that gcf (**H**) converges, that is

$$K = \lim_{n\to\infty} A_n B_n^{-1}.$$

Define $Y_n = A_n - KB_n$ and $Z_n = B_n$. Then we have $Z_n \in \mathcal{R}^*$ for almost all n,

$$Y_n Z_n^{-1} = (A_n - KB_n)B_n^{-1} = A_n B_n^{-1} - K \to 0,$$

and

$$\begin{aligned} Y_{-1} &= A_{-1} - KB_{-1} = I \in \mathcal{R}^*, \\ Z_0 - Z_{-1}Y_{-1}^{-1}Y_0 &= B_0 - B_{-1}(A_0 - KB_0) = c_{10}^{-1} \in \mathcal{R}^* \end{aligned}$$

Finally, $Y_{-1} = I$ and $Y_0 = b_0 c_{10}^{-1} - Kc_{10}^{-1}$ yield $K = b_0 - Y_{-1}^{-1}Y_0 c_{10}$.

Now, let (Y_n) and (Z_n) fulfill the properties stated in the theorem. Then, for $n = -1, 0$, we can write

$$\begin{pmatrix} B_n \\ A_n \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} Y_n \\ Z_n \end{pmatrix} \tag{8.3.2}$$

where

$$\begin{aligned} \alpha &= -c_{10}^{-1}\left(Z_0 - Z_{-1}Y_{-1}^{-1}Y_0\right)^{-1} Z_{-1}Y_{-1}^{-1}, \\ \beta &= c_{10}^{-1}\left(Z_0 - Z_{-1}Y_{-1}^{-1}Y_0\right)^{-1}, \\ \gamma &= Y_{-1}^{-1} - \left(b_0 c_{10}^{-1} - Y_{-1}^{-1}Y_0\right)\left(Z_0 - Z_{-1}Y_{-1}^{-1}Y_0\right)^{-1} Z_{-1}Y_{-1}^{-1}, \\ \delta &= \left(b_0 c_{10}^{-1} - Y_{-1}^{-1}Y_0\right)\left(Z_0 - Z_{-1}Y_{-1}^{-1}Y_0\right)^{-1}. \end{aligned}$$

Since $(A_n), (B_n), (Y_n), (Z_n)$ fulfill (8.3.1) for $n \geq 1$, by induction, we obtain that (8.3.2) is valid for all $n \in \mathbb{N}$. For sufficiently large n, we obtain

$$A_n B_n^{-1} = (\gamma Y_n + \delta Z_n)(\alpha Y_n + \beta Z_n)^{-1} = \left(\gamma Y_n Z_n^{-1} + \delta\right)\left(\alpha Y_n Z_n^{-1} + \beta\right)^{-1}$$

where $\alpha Y_n Z_n^{-1} + \beta \in \mathcal{R}^*$ for sufficiently large n due to convergence to $\beta \in \mathcal{R}^*$. Furthermore, we obtain

$$\begin{aligned} \lim_{n\to\infty} A_n B_n^{-1} &= \delta\beta^{-1} \\ &= \left(b_0 c_{10}^{-1} - Y_{-1}^{-1}Y_0\right)\left(Z_0 - Z_{-1}Y_{-1}^{-1}Y_0\right)^{-1}\left(c_{10}^{-1}\left(Z_0 - Z_{-1}Y_{-1}^{-1}Y_0\right)^{-1}\right)^{-1} \\ &= \left(b_0 c_{10}^{-1} - Y_{-1}^{-1}Y_0\right) c_{10} = b_0 - Y_{-1}^{-1}Y_0 c_{10} \end{aligned}$$

□

Remark 8.3.1. In case of $\mathcal{R} = \mathbb{C}^{d\times d}$, we can interprete

$$M = \begin{pmatrix} Y_{-1} & Y_0 \\ Z_{-1} & Z_0 \end{pmatrix} \in \mathbb{C}^{2d\times 2d}.$$

Then, the assumption $Y_{-1}, Z_0 - Z_{-1}Y_{-1}^{-1}Y_0 \in \mathcal{R}^*$ can be rewritten as 'Let Y_{-1} and M be invertible'. This condition corresponds to that for LB-fractions (see section 7.4) in [LB96].

Next, we turn to convergence of $B_n^{(1)}B_n^{-1}$. Here, (8.3.1) is valid for $n \geq 2$, so in principle, we have to shift the indices by one. Unfortunately, when only shifting indices, we would result in a statement concerning convergence for $B_n \left(B_n^{(1)}\right)^{-1}$. When 'repairing' this problem, we have to make some more invertibility assumptions.

Theorem 8.3.2. *$B_n^{(1)}B_n^{-1}$ converges if and only if there are solutions $(Y_n), (Z_n)$ for*

$$X_n c_{n+1,n} = X_{n-1}b_n + \sum_{m=1}^{n-1} X_{m-1}a_{mn}, \qquad n = 2, 3, \ldots \tag{8.3.3}$$

with

- $Z_0, Y_1 - Y_0Z_0^{-1}Z_1, Y_0b_1 - Y_1c_{21}, I - \left(b_1c_{21}^{-1} - Z_0^{-1}Z_1\right)\left(Y_1 - Y_0Z_0^{-1}Z_1\right)^{-1}Y_0 \in \mathcal{R}^*$,
- $Z_n \in \mathcal{R}^*$ *for almost all* n,
- $\lim\limits_{n\to\infty} Y_nZ_n^{-1} = 0$.

In this case, we have

$$L = \lim_{n\to\infty} B_n^{(1)}B_n^{-1} = (Y_0b_1 - Y_1c_{21})^{-1}Y_0c_{10}.$$

Proof. First, we assume again that $B_n^{(1)}B_n^{-1}$ converges to some limit L, we define $Y_n = B_n^{(1)} - LB_n$ and $Z_n = B_n$, and obtain

$$\lim_{n\to\infty} Y_nZ_n^{-1} = \lim_{n\to\infty} B_n^{(1)}B_n^{-1} - L = 0$$

and

$$\begin{aligned}
Z_0 &= c_{10}^{-1} \in \mathcal{R}^*, \\
Y_1 - Y_0Z_0^{-1}Z_1 &= c_{21}^{-1} - Lc_{10}^{-1}b_1c_{21}^{-1} + Lc_{10}^{-1}c_{10}c_{10}^{-1}b_1c_{21}^{-1} = c_{21}^{-1} \in \mathcal{R}^*, \\
Y_0b_1 - Y_1c_{21} &= -Lc_{10}^{-1}b_1 - \left(c_{21}^{-1} - Lc_{10}^{-1}b_1c_{21}^{-1}\right)c_{21} = -I \in \mathcal{R}^*, \\
I - \left(b_1c_{21}^{-1} - Z_0^{-1}Z_1\right)\cdot\ldots &= I - 0 = I \in \mathcal{R}^*.
\end{aligned}$$

Now let Y and Z fulfill the assumptions of the Theorem. Similar to the proof for Theorem 8.3.1, we write

$$\begin{pmatrix} B_n \\ B_n^{(1)} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} Y_n \\ Z_n \end{pmatrix} \tag{8.3.4}$$

where

$$\begin{aligned}
\alpha &= c_{10}^{-1}\left(b_1c_{21}^{-1} - Z_0^{-1}Z_1\right)\left(Y_1 - Y_0Z_0^{-1}Z_1\right)^{-1}, \\
\beta &= c_{10}^{-1}Z_0^{-1} - c_{10}^{-1}\left(b_1c_{21}^{-1} - Z_0^{-1}Z_1\right)\left(Y_1 - Y_0Z_0^{-1}Z_1\right)^{-1}Y_0Z_0^{-1}, \\
\gamma &= c_{21}^{-1}\left(Y_1 - Y_0Z_0^{-1}Z_1\right)^{-1}, \\
\delta &= -c_{21}^{-1}\left(Y_1 - Y_0Z_0^{-1}Z_1\right)^{-1}Y_0Z_0^{-1}.
\end{aligned}$$

For $n = 0$ and $n = 1$, (8.3.4) can be verified directly, for $n \geq 2$ it follows inductively since $(B_n), \left(B_n^{(1)}\right), (Y_n), (Z_n)$ satisfy (8.3.3). Similar to the proof for theorem 8.3.1, we obtain

$$\lim_{n\to\infty} B_n^{(1)} B_n^{-1} = \delta\beta^{-1} = -c_{21}^{-1} \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0 \left(I - \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right) \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0\right)^{-1} c_{10},$$

and the representation of L then follows from

$$\begin{aligned}
& -c_{21}^{-1} \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0 \left(I - \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right) \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0\right)^{-1} c_{10} \\
& \quad -(Y_0 b_1 - Y_1 c_{21})^{-1} Y_0 c_{10} \\
= \; & \left(-c_{21}^{-1} \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0 - (Y_0 b_1 - Y_1 c_{21})^{-1} Y_0 \left(I - \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right) \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0\right)\right) \\
& \quad \cdot \left(I - \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right) \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0\right)^{-1} c_{10} \\
= \; & \left(-c_{21}^{-1} - (Y_0 b_1 - Y_1 c_{21})^{-1} \left(Y_1 - Y_0 Z_0^{-1} Z_1\right) + (Y_0 b_1 - Y_1 c_{21})^{-1} Y_0 \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right)\right) \\
& \quad \cdot \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0 \left(I - \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right) \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0\right)^{-1} c_{10} \\
= \; & (Y_0 b_1 - Y_1 c_{21})^{-1} \left(-(Y_0 b_1 - Y_1 c_{21}) c_{21}^{-1} - \left(Y_1 - Y_0 Z_0^{-1} Z_1\right) + Y_0 \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right)\right) \\
& \quad \cdot \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0 \left(I - \left(b_1 c_{21}^{-1} - Z_0^{-1} Z_1\right) \left(Y_1 - Y_0 Z_0^{-1} Z_1\right)^{-1} Y_0\right)^{-1} c_{10} = 0.
\end{aligned}$$

□

In Theorem 8.3.2, we characterized the convergence of $L_1^{(n)} := B_n^{(1)} B_n^{-1}$, which is equivalent to convergence of $w_1^{(n)}(w_0, \mathbf{H})$ for all choices w_0. Naturally, a generalization consists in considering $\lim_{n\to\infty} B_n^{(\ell)} B_n^{-1}$ for some $\ell \geq 2$ or for all $\ell \in \mathbb{N}$. Finding easy-to-handle conditions requires further research.

8.4 Literature review

Concerning Miller's algorithm, we already referred to literature, see section 8.1. For many special cases, Pincherle–type convergence criteria can be found in the literature:

- For non-generalized continued fractions in $\mathbb{C}$, we refer to [Gau67; Gau72].
- For n-fractions in $\mathbb{C}$, Pincherle–type criteria were given by van der Cruyssen [Cru79; Cru82].
- Hanschke [Han98] generalized the definition of dominance ('σ-dominance'), and considered vector-valued recurrence schemes, see section 7.4. In this context, he considered adjoint systems, and he proved a Pincherle–type convergence criterion.
- Levrie and Bultheel proved a Pincherle–type criterion for LB-fracions, see [LB96].

Chapter 9

Periodic GCFs

If in $\mathbb{C}$, the periodic continued fraction

$$K = b + \cfrac{ac}{b + \cfrac{ac}{b + \ddots}}$$

converges with value $K \neq 0$, K fulfills the equation $x = b + \frac{ac}{x}$, that is, K is a root of $x^2 - bx - ac = 0$. Furthermore, if x_1, x_2 are the roots of $x^2 - bx - ac = 0$, K converges if and only if $x_1 = x_2$ or $|x_1| > |x_2|$. In this case, $K = x_1$, that is the continued fraction characterizes the maximal solution of a quadratic equation, see [Per57, Theorem 2.38]. Conversely, $K \neq 0$ and $K = b + \frac{ac}{K}$ yield that $L := \frac{c}{K}$ is a root of $ay^2 + by - c$, and it follows that L converges to the minimal root of $ay^2 + by - c$.

Non-generalized continued fractions can be obtained from gcfs in the sense of chapter 7 by setting $a_{mn} = 0$ for $n \geq m + 2$. If we allow $a_{mn} \neq 0$ for $n \geq m + 2$, but still consider an appropriate kind of 'periodicity' of the coefficients, it will turn out that the quadratic polynomial $ay^2 + by - c$ is replaced by a power series.

In section 9.1, we will define periodicity and we will give a sufficient criterion for $L := L_{1,1,0}$ to be a root of the function defined by the corresponding power series. In $\mathbb{C}$, the characterization of this root as the minimal one is still valid, we will prove this fact in section 9.2.

Finally, in sections 9.3 and 9.4, we will consider a matrix-valued power series in the context of Markov chains with a special structure. In combination with Neuts matrix-geometric methods [Neu81], we will present a derivation of the waiting time distribution for servers with deterministic service times and geometrically distributed interarrival times. By considering an appropriate limit, we will obtain the waiting time distribution for Poisson input, that is, for the $M/D/1$-queueing model. Although the result is not new, it seems reasonable to believe that the method can be extended to other problems where no exact results are known.

9.1 Periodicity

In this chapter, we generally assume $b_0 = 0$, $a_{0m} = 0$, yielding $\operatorname{gcf}(\mathbf{H}) = 0$ whenever $\operatorname{gcf}(\mathbf{H})$ converges. Hence, the initial condition in $\mathbf{H}x = 0$ becomes trivial. Periodicity of a non-

generalized continued fraction is given by

$$c_{n,n-1} = -\alpha_0,\ b_n = \alpha_1,\ a_{n,n-1} = \alpha_2,$$

and in the upper-Hessenberg-case we use the natural generalization

$$c_{n,n-1} = -\alpha_0,\ b_n = \alpha_1,\ a_{nm} = \alpha_{m-n+1}, \qquad n \geq 1.$$

In this case, we have $B_N^{(\ell)} = B_{N-\ell}$, and thus, if

$$L := \lim_{N\to\infty} B_N^{(1)} B_N^{-1} = \lim_{N\to\infty} B_{N-1} B_N^{-1}$$

converges, we have

$$L = \lim_{N\to\infty} B_N^{(n)} \left(B_N^{(n-1)}\right)^{-1} \qquad \text{and thus} \qquad \lim_{N\to\infty} B_N^{(n)} B_N^{-1} = L^n$$

for all $n \in \mathbb{N}$, implying $w_n(w_0, \mathbf{H}) = L^n w_0$. Assuming that $w(I, \mathbf{H})$ solves $\mathbf{H}x = 0$ (this is not always true, see Example 4.2.1), we obtain

$$-\alpha_0 L^{n-1} = \sum_{m=1}^{\infty} \alpha_m L^{m+n-1}, \qquad n \geq 1,$$

which is equivalent to

$$0 = \sum_{m=0}^{\infty} \alpha_m L^m.$$

These considerations motivate

Definition 9.1.1. Let $\alpha = (\alpha_n)_{n\in\mathbb{N}_0}$ be a $\mathcal{R}$-valued sequence with $\alpha_0 \in \mathcal{R}^*$. Define $(B_n)_{n\in\mathbb{N}_0}$ by

$$B_0\alpha_0 + I = 0, \qquad \sum_{m=0}^{n} B_m \alpha_{n-m} = \sum_{m=0}^{n} B_{n-m}\alpha_m = 0, \qquad n \in \mathbb{N}. \tag{9.1.1}$$

If $B_n \in \mathcal{R}^*$ for almost all $n \in \mathbb{N}$, and if

$$L = \lim_{N\to\infty} B_{N-1} B_N^{-1}$$

exists, L is said to be a convergent periodic inverse generalized continued fraction, abbreviated by $L = \operatorname{pig}(\alpha)$.

In the literature, equations of the form (9.1.1) are referred to as *Volterra difference equation of convolution type*, see [Ela09]. In contrast to the general recursion for the B_n, in case of periodic coefficients, the multiplication from the right can be replaced by multiplications from the left.

Lemma 9.1.1. *Let $(\alpha_n)_{n\in\mathbb{N}_0}$ be a $\mathcal{R}$-valued sequence with $\alpha_0 \in \mathcal{R}^*$, and let the sequence $(B_n)_{n\in\mathbb{N}_0}$ be defined by* (9.1.1)*. Then we have*

$$\alpha_0 B_0 + I = 0, \qquad \sum_{m=0}^{n} \alpha_{n-m} B_m = \sum_{m=0}^{n} \alpha_m B_{n-m} = 0, \qquad n \in \mathbb{N}. \tag{9.1.2}$$

Proof. Obviously, both (9.1.1) and (9.1.2) yield $B_0 = -\alpha_0^{-1}$ and $B_1 = \alpha_0^{-1}\alpha_1\alpha_0^{-1}$. By induction, we then obtain

$$\begin{aligned}
B_n &\overset{(9.1.1)}{=} -B_0\alpha_n\alpha_0^{-1} - \sum_{m=1}^{n-1} B_m\alpha_{n-m}\alpha_0^{-1} \\
&= \alpha_0^{-1}\alpha_n\alpha_0^{-1} - \sum_{m=0}^{n-2} B_{m+1}\alpha_{n-1-m}\alpha_0^{-1} \\
&\overset{(9.1.2) \text{ for } 1,\dots,n-1}{=} -\alpha_0^{-1}\alpha_n B_0 + \sum_{m=0}^{n-2} \alpha_0^{-1} \sum_{k=1}^{m+1} \alpha_k B_{m+1-k}\alpha_{n-1-m}\alpha_0^{-1} \\
&= -\alpha_0^{-1}\alpha_n B_0 + \alpha_0^{-1} \sum_{k=1}^{n-1} \alpha_k \sum_{m=k-1}^{n-2} B_{m+1-k}\alpha_{n-1-m}\alpha_0^{-1} \\
&\overset{\ell=m+1-k}{=} -\alpha_0^{-1}\alpha_n B_0 + \alpha_0^{-1} \sum_{k=1}^{n-1} \alpha_k \sum_{\ell=0}^{n-1-k} B_\ell\alpha_{n-k-\ell}\alpha_0^{-1} \\
&\overset{(9.1.1)}{=} -\alpha_0^{-1}\alpha_n B_0 - \alpha_0^{-1} \sum_{k=1}^{n-1} \alpha_k B_{n-k} \\
&= -\alpha_0^{-1} \sum_{k=0}^{n-1} \alpha_k B_{n-k},
\end{aligned}$$

for all n. □

Theorem 9.1.1. *Let $(\alpha_n)_{n\in\mathbb{N}_0}$ be a $\mathcal{R}$-valued sequence with $\alpha_0 \in \mathcal{R}^*$, let all approximants for $L = \operatorname{pig}(\alpha)$ be well-defined, and let L converge. Furthermore, let*

$$\sum_{n=0}^{\infty} ||\alpha_n||\, x^n$$

converge in a neighbourhood of $||L||$. Define $F_\alpha : D_F \to \mathcal{R}$ by

$$F_\alpha(Z) = \sum_{m=0}^{\infty} \alpha_m Z^m, \qquad D_F := \{Z \in \mathcal{R} : \ \sum ||\alpha_m Z^m|| < \infty\}.$$

Then $L \in D_F$ and $F_\alpha(L) = 0$.

Proof. $||\alpha_m L^m|| \leq ||\alpha_m|| \cdot ||L||^m$ implies $L \in D_F$. For proving $F_\alpha(L) = 0$, we use Lemma 9.1.1. Multiplying (9.1.2) by B_n^{-1} yields

$$\sum_{m=0}^{n} \alpha_m B_{n-m} B_n^{-1} = 0,$$

or equivalently

$$\alpha_0 + \sum_{m=1}^{n} \alpha_m \prod_{r=n-m+1}^{n} B_{r-1} B_r^{-1}. \tag{9.1.3}$$

For fixed m and $n \to \infty$, the product $\prod_{r=n-m+1}^{n} B_{r-1} B_r^{-1}$ converges to L^m. The statement now follows by the dominated convergence theorem: Define $M = \max\left\{1, \sup_{n\in\mathbb{N}} \frac{||B_{n-1}B_n^{-1}||}{||L||}\right\}$, choose $\epsilon > 0$ and $N_0 \in \mathbb{N}_0$ such that the power series converges in $(1+\epsilon)\,||L||$ and $\frac{||B_{n-1}B_n^{-1}||}{||L||} \leq 1+\epsilon$ for all $n \geq N_0$. Taking

$$\left|\left| \prod_{r=n-m+1}^{n} B_{r-1} B_r^{-1} \right|\right| \leq ||L||^m \cdot (1+\epsilon)^m \cdot M^{N_0}$$

as the convergent majorant yields the result. □

Next, we want to specify our Pringsheim-type criteria for periodic gcfs. Here, we use $L = L_{1,1,0} = -K_1^{-1}\alpha_0$ and apply Theorem 7.3.2, or we use $L = w_1$ in Theorem 7.3.3.

Theorem 9.1.2. *Let $(\alpha_n)_{n\in\mathbb{N}_0}$ be a $\mathcal{R}$-valued sequence with $\alpha_0, \alpha_1 \in \mathcal{R}^*$, $\alpha_n \neq 0$ for some $n \geq 2$, $\lambda > 0$ and*

$$\left|\left|\alpha_0\alpha_1^{-1}\right|\right| + \sum_{m=2}^{\infty} \lambda^m \left|\left|\alpha_m\alpha_1^{-1}\right|\right| \leq \lambda. \tag{9.1.4}$$

Then $L = \operatorname{pig}(\alpha)$ converges with $L \in \mathcal{R}^*$ and

$$\left|\left|\left(\alpha_0 L^{-1} + \alpha_1\right)\rho\right|\right| \leq \frac{1}{\lambda}\,||\alpha_0\rho|| \tag{9.1.5}$$

for all $\rho \in \mathcal{R}^*$.

Proof. With $c_{n+1,n} = -\alpha_0$, $b_n = \alpha_1$, $a_{mn} = \alpha_{n+1-m}$ and $\lambda_n = \frac{1}{\lambda^n} \cdot I$, the conditions of Theorem 7.3.3 simplify to

$$\frac{1}{\lambda}\left|\left|\alpha_0\alpha_1^{-1}\right|\right| \leq 1$$

and

$$\frac{1}{\lambda}\left|\left|\alpha_0\alpha_1^{-1}\right|\right| + \sum_{m=1}^{n-1} \lambda^{n-m}\left|\left|\alpha_{n+1-m}\alpha_1^{-1}\right|\right| = \frac{1}{\lambda}\left|\left|\alpha_0\alpha_1^{-1}\right|\right| + \sum_{m=2}^{n} \lambda^{m-1}\left|\left|\alpha_m\alpha_1^{-1}\right|\right| \leq 1$$

for all $n \geq 2$, where at least one inequality shall be strict. This is guaranteed by (9.1.4) and $\alpha_n \neq 0$ for some $n \geq 2$. Therefore, L converges.

Using the same argumentation, according to Theorem 7.3.2, $K_1 = \lim K_1^{(N)}$ converges with

$$||\lambda_1(K_1 - b_1)\rho_1|| \leq ||\lambda_2 c_{21}\rho_1||,$$

and since $L_1^{(N)} = \left(K_1^{(N)}\right)^{-1} c_{10}$, we have $L_1 = -K_1^{-1}\alpha_0$, and thus, $L_1 \in \mathcal{R}^*$, and $K_1 = -\alpha_0 L_1^{-1}$ yields (9.1.5). □

Remark 9.1.1. We derived Theorem 9.1.2 from Theorem 7.3.2 and Theorem 7.3.3 by specifying $c_{n+1,n} = -\alpha_0$, $b_n = \alpha_1$, $a_{mn} = \alpha_{n+1-m}$ and $\lambda_n = \frac{1}{\lambda^n} \cdot I$ with some $\lambda \in \mathbb{R}_{>0}$. The reason for this specification of λ_n is that it results in **one** simple condition (9.1.4), whereas more general choices would yield more than one condition.

We give an easy example for the application of Theorem 9.1.2 in which we will obtain a gcf-representation for the minimal root of an entire function in $\mathbb{C}$.

Example 9.1.1. We consider the function $f : \mathbb{C} \to \mathbb{C}$, defined by

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-z)^n.$$

Obviously, f is an entire function fulfilling $f(z) = \cos(\sqrt{z})$ for $z \in \mathbb{R}_{\geq 0}$ and $f(z) = \cosh(\sqrt{-z})$ for $z \in \mathbb{R}_{\leq 0}$. The roots of f are

$$\frac{\pi^2}{4}, \frac{9\pi^2}{4}, \frac{25\pi^2}{4}, \ldots.$$

Using the notations of Theorem 9.1.1, we have the coefficients $\alpha_n = \frac{(-1)^n}{(2n)!}$. Due to the fact that f is an entire function, Theorem 9.1.1 tells us that, in case of convergence, $\operatorname{pig}(\alpha)$ is a root of f. For proving convergence, we apply Theorem 9.1.2, where we remark that for $\mathcal{R} = \mathbb{C}$, (9.1.4) is equivalent to

$$|\alpha_0| - \lambda\,|\alpha_1| + \sum_{m=2}^{\infty} \lambda^m\,|\alpha_m| \leq 0.$$

Here, we compute

$$\begin{aligned} |\alpha_0| - \lambda\,|\alpha_1| + \sum_{m=2}^{\infty} \lambda^m\,|\alpha_m| &= 1 - \frac{\lambda}{2} + \sum_{m=2}^{\infty} \frac{\lambda^m}{(2m)!} = -\lambda + \sum_{m=0}^{\infty} \frac{\lambda^m}{(2m)!} \\ &= -\lambda + \cosh(\sqrt{\lambda}), \end{aligned}$$

and it is easy to find some $\lambda > 0$ with $-\lambda + \cosh\sqrt{\lambda} \leq 0$, e.g. $\lambda = 4$. Thus, by Theorem 9.1.2, $L = \operatorname{pig}(\alpha)$ converges and for its value, we have $\left|L^{-1} - \frac{1}{2}\right| \leq \frac{1}{4}$. Since all coefficients are elements of $\mathbb{R}$, $L, L^{-1} \in \mathbb{R}$, and therefore $L^{-1} \in \left[\frac{1}{4}, \frac{3}{4}\right]$, yielding $L \in \left[\frac{4}{3}, 4\right]$. Since we know that there is just one root of f in this interval, we can state

$$\operatorname{pig}(\alpha) = \frac{\pi^2}{4}.$$

Obviously, this is the minimal root of f.

In this example, we have seen $\operatorname{pig}(\alpha)$ converging to the minimal root of the entire function $\sum \alpha_n z^n$, and indeed, for $\mathcal{R} = \mathbb{C}$, we will prove a quite general result in this direction.

9.2 Periodic GCFs in $\mathbb{C}$

In this section, we will investigate whether convergence and value of $\operatorname{pig}(\alpha)$ can be obtained from the structure of the set of roots of the analytic function $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, independent on whether a Pringsheim-type criterion is fulfilled.

We begin with a fundamental result concerning meromorphic functions.

Theorem 9.2.1. *Let the complex-valued function f be defined meromorphically on $U_\sigma(0) = \{z \in \mathbb{C} : \ |z| < \sigma\}$ for some $\sigma > 0$, with neither root nor pole in 0, and let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ for sufficiently small $|z|$. Furthermore, let $z_1, z_2, \ldots$ be all roots of f in $U_\sigma(0)$ with multiplicities $m_1, m_2, \ldots$, where either $|z_i| < |z_{i+1}|$ or $|z_i| = |z_{i+1}|$ and $m_i \geq m_{i+1}$.*

Under these conditions, $L = \operatorname{pig}(\alpha)$ converges with value in $U_\sigma(0)$ if and only if f has at least one root in $U_\sigma(0)$ and if – in the case of more than one root – $|z_1| < |z_2|$ or $m_1 > m_2$. In this case, $L = z_1$.

Proof. Note that f may have infinitely many roots or poles in $U_\sigma(0)$, but neither roots nor poles have an accumulation point within $U_\sigma(0)$. Thus, for all $r < \sigma$, f has finitely many roots and poles in $U_r(0)$.

Define $B(z) = \sum_{n=0}^{\infty} B_n z^n$, that is, B is the generating function of the sequence $(B_n)_{n\geq 0}$ defined by (9.1.1). We have

$$B(z)f(z) = \sum_{n=0}^{\infty} B_n z^n \sum_{m=0}^{\infty} \alpha_m z^m = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(B_{n-m}\alpha_m\right) z^n = -1,$$

yielding the important equation

$$B(z) = -\frac{1}{f(z)}. \tag{9.2.1}$$

If we choose $r > 0$ such that f has neither roots nor poles in $U_r(0)$, (9.2.1) holds in $U_r(0)$, and, since f is a meromorphic function, the equation holds on the greatest disc around 0 in which B is holomorphic. Since poles of f become roots for B, and roots of f become poles for B (the multiplicities are preserved), this greatest disc is $U_\sigma(0)$ if f has no roots in $U_\sigma(0)$, and it is $U_{|z_1|}(0)$ if f has at least one root. Now we come back to

$$L = \operatorname{pig}(\alpha) = \lim_{n\to\infty} \frac{B_n}{B_{n+1}}.$$

By (9.2.1), the question whether L converges to a (minimal) root of f is transformed into the question whether the quotient of coefficients of a power series converges to a (minimal) pole.

Now, the derivation of the statement of the Theorem is 'standard' complex analysis. Actually, it could be derived from statements concerning the asymptotic behaviour of the coefficients of power series, see [FS09]. Nevertheless, we give a direct proof here.

Obviously, if L converges, $|L|$ is the radius of convergence of $B(z)$. This has three immediate consequences:

- $L \neq 0$.
- If $L \notin U_\sigma(0)$, B has no pole in $U_\rho(0)$, implying that f has no root in $U_\sigma(0)$.
- If $L \in U_\sigma(0)$, there is no pole of B with absolute value smaller than $|L|$.

Now, assume that f has roots in $U_\sigma(0)$. Furthermore, assume that $z_1, \ldots, z_M$ have the same absolute value $< \sigma$ and that $z_1, \ldots, z_{M'}$ have the same multiplicity ($M' \leq M$). We have to prove that $L = z_1$ for $M' = 1$ and that L does not converge for $M' > 1$. Using Laurent series, we can write

$$B(z) = \sum_{n=0}^{m_1-1} h_n^{(1)} (z_1 - z)^{-(n+1)} + R_1(z),$$

where $R_1(z)$ is holomorphic in $U_\sigma(0) \setminus \{z_2, z_3, \ldots\}$. Similarly, we have

$$R_1(z) = \sum_{n=0}^{m_2-1} h_n^{(2)} (z_2 - z)^{-(n+1)} + R_2(z),$$

where $R_2(z)$ is holomorphic in $U_\sigma(0) \setminus \{z_3, z_4, \ldots\}$. Finally, this method results in

$$B(z) = \sum_{k=1}^{M} \sum_{n=0}^{m_k-1} h_n^{(k)} (z_k - z)^{-(n+1)} + R(z) = H(z) + R(z),$$

where $R(z)$ is holomorphic on $U_\rho(0) \setminus \{z_{M+1}, z_{M+2}, \ldots\}$. Writing $R(z) = \sum_{n=0}^{\infty} r_n z^n$ therefore implies $\limsup \sqrt[n]{|r_n|} < \frac{1}{|z_1|}$. Now, for $H(z) = \sum_{n=0}^{\infty} h_n z^n$, we have

$$H(z) = \sum_{k=1}^{M} \sum_{n=0}^{m_k-1} h_n^{(k)} (z_k - z)^{-(n+1)} = \sum_{k=1}^{M} \sum_{n=0}^{m_k-1} h_n^{(k)} \sum_{j=0}^{\infty} \binom{n+j}{n} \frac{z^j}{z_k^{n+1+j}},$$

implying (due to $|z_1| = \ldots = |z_M|$ and $m_1 = \ldots = m_{M'} > m_{M'+1} \geq \ldots \geq m_M$)

$$\begin{aligned} h_j \cdot z_1^j &= z_1^j \cdot \sum_{k=1}^{M} \sum_{n=0}^{m_k-1} \frac{h_n^{(k)}}{z_k^{n+1+j}} \binom{j+n}{n} \\ &= \frac{1}{(m_1-1)!} \cdot \sum_{k=1}^{M'} \frac{h_{m_1-1}^{(k)}}{z_k^{m_1}} \cdot \left(\frac{z_1}{z_k}\right)^j \cdot j^{m_1-1} + O\left(j^{m_1-2}\right). \end{aligned}$$

Since $\left|\frac{z_1}{z_k}\right| = 1$ for $k = 1, \ldots, M$ and $\frac{z_1}{z_k} \neq 1$ for $k > M'$, we know that the fraction

$$\frac{h_j z_1^j}{h_{j+1} z_1^{j+1}}$$

is always bounded and converges if and only if $M' = 1$, and in this case we have convergence to 1, or, equivalently,

$$\lim_{j\to\infty} \frac{h_j}{h_{j+1}} = z_1.$$

Independently on whether or not we have convergence, we know that $\sqrt[n]{|h_n|} \to \frac{1}{|z_1|}$. Since from above we have $\limsup \sqrt[n]{|r_n|} < \frac{1}{|z_1|}$, we know $\frac{r_n}{h_n} \to 0$, and therefore (since $\frac{h_n}{h_{n+1}}$ is bounded) we derive

$$\lim_{n\to\infty} \frac{B_n}{B_{n+1}} = \lim_{n\to\infty} \frac{h_n + r_n}{h_{n+1} + r_{n+1}} = \lim_{n\to\infty} \frac{\frac{h_n}{h_{n+1}} + \frac{r_n}{h_n} \cdot \frac{h_n}{h_{n+1}}}{1 + \frac{r_{n+1}}{h_{n+1}}} = \lim_{n\to\infty} \frac{h_n}{h_{n+1}}.$$

Note that this result also yields that $B_n \neq 0$ for almost all $n \in \mathbb{N}_0$. □

In particular, the result guarantees that $L = \operatorname{pig}(\alpha)$ converges to the minimal root of a meromorphic function independent on whether or not $\sum \alpha_n L^n$ converges.

Example 9.2.1. Consider $f(z) = \frac{(z-2)(2-3)}{(1-z)}$. For $|z| < 1$, we have

$$f(z) = 6 + z + \sum_{n=2}^{\infty} 2z^n,$$

that is, we have $\alpha_0 = 6$, $\alpha_1 = 1$ and $\alpha_n = 2$ for all $n \geq 2$. According to Theorem 9.2.1, $\operatorname{pig}(\alpha)$ converges to 2, although the power series representation of f diverges in 2.

Since the equation $B(z) = -\frac{1}{f(z)}$ was the key for the proof of Theorem 9.2.1, this fact does not surprise: If f has a pole at z_0, B is analytic in z_0 with $B(z_0) = 0$. In particular, Theorem 9.2.1 states that periodic n-fractions characterize minimal roots of polynomials of degree $n + 1$. In contrasts to polynomials, general functions (analytic in $U_\epsilon(0)$) may have essential singularities. And if f has an essential singularity in z_0, so has $B = -\frac{1}{f}$. Since these singularities restrict the radius of convergence of $B(z)$, it does not surprise that $\operatorname{pig}(\alpha)$ may converge to essential singularities of f instead of roots. Actually, the main question is whether an essential singularity or a pole dominates the asymptotic behaviour of the coefficients of the power series expansion of $B(z)$. Here, more general results concerning the asymptotic behaviour of these coefficients are helpful. For such results, we again refer to [FS09], and omit further considerations.

An idea is combining the Pringsheim-type criterion 9.1.2 for proving convergence of $\operatorname{pig}(\alpha)$ and Theorem 9.2.1. For example, one might use Pringsheim-type criteria for proving that a function has a unique minimal root of maximal order. This approach leads to

Theorem 9.2.2. *Let the complex valued function f be meromorphic on $\mathbb{C}$, let*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

for small $|z|$, let $\alpha_0, \alpha_1 \neq 0$, let $\alpha_n \neq 0$ for some $n \geq 2$, let $\lambda > 0$, and let

$$|\alpha_0| - \lambda |\alpha_1| + \sum_{m=2}^{\infty} \lambda^m |\alpha_m| \leq 0 \tag{9.2.2}$$

hold. Then f has a unique minimal root of maximal order z_0 for which we have $\operatorname{pig}(\alpha) = z_0$ *and*

$$\left| \frac{\alpha_0}{z_0} + \alpha_1 \right| \leq \frac{|\alpha_0|}{\lambda}. \tag{9.2.3}$$

Proof. For $\mathcal{R} = \mathbb{C}$, (9.2.2) and (9.1.4) are equivalent, the same is true for (9.2.3) and (9.1.5). The fact that $\operatorname{pig}(\alpha)$ converges is obtained from Theorem 9.1.2, and due to Theorem 9.2.1, this root is minimal of maximal order. □

Finally, we want to demonstrate how to use the gcf-based method for determining all roots of a non-constant complex-valued function f which we assume to be meromorphic on $\mathbb{C}$. Applying our method to

$$f_x(z) = f(x+z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} z^n,$$

that is, $\alpha_n = \frac{f^{(n)}(x)}{n!}$, yields that if $L_x := \operatorname{pig}(\alpha)$ converges, it is the minimal root of f_x, implying that $x + L_x$ is a root of f. By varying x in an appropriate way, we are able to determine all roots of f.

For determining convergence, we can try to use Theorem 9.2.2, which guarantees convergence in case of

$$|f(x)| - \left|f'(x)\right| \lambda + \sum_{n=2}^{\infty} \frac{\left|f^{(n)}(x)\right|}{n!} \lambda^n \leq 0$$

for some λ which might depend on x.

Now let us assume that $f^{(n)}(x) \in \mathbb{R}$ for all $n \in \mathbb{N}_0$ and for all $x > x_0$, we have $f(x) > 0$. Furthermore, let us assume that there is $x_1 \geq x_0$ such that for all $x \geq x_1$, we have $f'(x) \geq 0$ and $f^{(n)}(x) \cdot (-1)^n \geq 0$ for all $n \geq 2$. Then, for all $x \geq x_1$, we have

$$\begin{aligned} & |f(x)| - \left|f'(x)\right| (x - x_0) + \sum_{m=2}^{\infty} \frac{\left|f^{(n)}(x)\right|}{n!} (x - x_0)^n \\ = \quad & f(x) + f'(x)(x_0 - x) + \sum_{m=2}^{\infty} \frac{f^{(n)}(x)}{n!} (x_0 - x) = f(x_0) = 0. \end{aligned}$$

Theorem 9.2.2 guarantees convergence of $\operatorname{pig}(\alpha)$ to the minimal root, that is the root with minimal distance to x. Since $\operatorname{pig}(\alpha) \in \mathbb{R}$, this root ist x_0. These considerations yield

Theorem 9.2.3. *Let*

- $x_0, x_1 \in \mathbb{R}$, $x_1 \geq x_0$,

- $f : G \to \mathbb{C}$ *holomorphic on* $H_{x_0} := \{z : \operatorname{Re}(z) > x_0\} \subset G$ *and continuous on* $H_{x_0} \cup \{x_0\} \subset G$ *with*
$$f(x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x_0 - x)^n.$$
- $f(x_0) = 0$, $f(x) > 0$ *for* $x > x_0$,
- $f'(x) \geq 0$ *for* $x \geq x_1$ *and* $(-1)^n f^{(n)}(x) \geq 0$ *for* $x \geq x_1$ *and* $n \geq 2$.

Then $\operatorname{Re}(z) \leq x_0$ *for every root* z *of* f.

If f is holomorphic in x_0, $f(x_0) = \ldots$ is trivially fulfilled. An easy example is given by
$$f(z) = z \ln z.$$
Here, we can choose $x_0 = x_1 = 1$, since $f'(z) = \ln z + 1 \geq 0$ for $z \geq 1$ and
$$(-1)^n \cdot f^{(n)}(z) = \frac{(n-2)!}{z^{n-1}} \geq 0$$
for $n \geq 2$ and $z \geq 1$. Therefore, $f(z) = z \ln z$ has no root with real part > 1.

Further examples are given by $f(z) = (z - x_0)^\alpha$ for $\alpha \in [1, 2]$. For $\alpha \in \{1, 2\}$, the conditions of Theorem 9.2.3 are trivially fulfilled. For $\alpha \in (1, 2)$, the constraints concerning the signs of the derivatives hold, but f has an essential singularity in x_0. Nevertheless, since the power series
$$(z - x_0)^\alpha = (-x_0)^\alpha \sum_{m=0}^{\infty} \binom{\alpha}{m} \left(-\frac{z}{x_0}\right)^m$$
converges to 0 for $z = x_0$, we can still apply Theorem 9.2.3, and thus, there are no roots of $(z - x_0)^\alpha$ with real part $> x_0$.

Although these examples are quite trivial, Theorem 9.2.3 may prove new statements concerning half–plains without roots.

The results for roots of analytic functions yield some results for sum equations with constant coefficients. These are based on

Theorem 9.2.4. *Let* $\mathcal{R} = \mathbb{C}$, *let* $\alpha_0 \neq 0$, *let* $f(z) = \sum\limits_{m=0}^{\infty} \alpha_m z^m$ *be holomorphic on* $G \supset H_\sigma := \{z : |z| \leq \sigma\}$, *let* $z_1, z_2, \ldots, z_k$ *be the roots of* f *in* H_σ, *with corresponding multiplicities* $m_1, m_2, \ldots, m_k$. *Then there are exactly* $m_1 + \ldots + m_k$ *linearly independent solutions* $(x_n)_{n \in \mathbb{N}_0}$ *for the sum equation*

$$0 = \sum_{m=0}^{\infty} \alpha_m x^{n+m}, \qquad n \in \mathbb{N}_0 \tag{9.2.4}$$

with $\limsup_{n\to\infty} \sqrt[n]{|x_n|} \leq \sigma$, *given by*
$$x_n = n^\ell z_j^n$$
for $j \in \{1, \ldots, k\}$ *and* $\ell \in \{0, \ldots, m_j - 1\}$.

Proof. See [Per21] □

Together with Theorem 9.2.4, Theorem 9.2.1 directly yields

Theorem 9.2.5. *Let $\mathcal{R} = \mathbb{C}$, let $f(z) = \sum_{m=0}^{\infty} \alpha_m z^m$ be holomorphic in $U_\sigma(0)$, and let there be at least one root in $U_\sigma(0)$. If (9.2.4) has a subdominant solution (x_n), $L = \operatorname{pig}(\alpha)$ converges and $x_n = L^n x_0$. If $L = \operatorname{pig}(\alpha)$ converges, by $x_n = L^n x_0$, a solution (x_n) of (9.2.4) is defined, and this solution dominates no other solution.*

Proof. Let us again assume that $z_1, z_2, \ldots, z_k$ are the roots of f in $U_\sigma(0)$, where $|z_i| < |z_{i+1}|$ or $|z_i| = |z_{i+1}|$ and $m_i > m_{i+1}$. Theorem 9.2.4 tells us that there is a subdominant solution if and only if $|z_1| < |z_2|$. This guarantees $\operatorname{pig}(\alpha) = z_1$, and thus, by $x_n = L^n x_0$ the subdominant solution is defined. On the other hand, if $\operatorname{pig}(\alpha)$ converges, it converges to z_1. □

Note that in case of convergence of $\operatorname{pig}(\alpha)$, we cannot state that $x_n = L^n x_0$ yields a subdominant solution. For example, consider the difference equation

$$x_m - x_{m+1} - x_{m+2} + x_{m+3} = 0, \qquad m \in \mathbb{N}_0.$$

Here, we have $\alpha_0 = \alpha_3 = 1$ and $\alpha_1 = \alpha_2 = -1$, and $f(z) = 1 - z - z^2 + z^3 = (1-z)^2(1+z)$ has the roots 1 (multiplicity 2) and -1. Therefore, three linearly independent solutions of the difference equation are given by (1), (n) and $((-1)^n)$, and there is no unique subdominant solution. Nevertheless, $\operatorname{pig}(\alpha) = 1$, and thus, the gcf-method yields the solution (x_n) with $x_n = x_0$, that is, a multiple of (1).

9.3 Periodic GCFs in $\mathbb{C}^{d\times d}$

Up to now, we were interested in characterizing the minimal root of some meromorphic function. Now, we consider the problem of characterizing the d minimal roots by means of periodic gcfs. However, here we focus on a special case which arises in the context of Markov chains. Since we are interested in computing an invariant measure, we consider the system $\psi\mathbf{P} = \psi$ instead of a system of the form $\mathbf{H}x = 0$. Due to this transposition, we consider lower Hessenberg matrices with repetitive structures instead of upper Hessenberg matrices.

Theorem 9.3.1. *Let $(p_n)_{n\in\mathbb{N}_0}$ be a sequence of non-negative numbers and let $(F_n)_{n\in\mathbb{N}_0}$ and*

$(G_n)_{n\in\mathbb{N}_0}$ *be sequences of* $\mathbb{R}^{d\times d}$*-matrices such that*

$$\begin{aligned}
\sum_{k=0}^{\infty} p_k &= 1, \qquad \textit{and} \qquad \sum_{k=0}^{\infty} k p_k > d,\\
G_0 &= \begin{pmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & & \ddots & \\ p_{d-1} & p_{d-2} & \cdots & p_0 \end{pmatrix},\\
G_n &= \begin{pmatrix} p_{nd} & p_{nd-1} & \cdots & p_{(n-1)d+1} \\ p_{nd+1} & p_{nd} & \cdots & p_{(n-1)d+2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{(n+1)d-1} & p_{(n+1)d-2} & \cdots & p_{nd} \end{pmatrix}, \quad n \geq 1, \qquad \textit{and}\\
\mathbf{P} &= \begin{pmatrix} F_0 & G_0 & & & \\ F_1 & G_1 & G_0 & & \\ F_2 & G_2 & G_1 & G_0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \qquad \textit{is stochastic and irreducible.}
\end{aligned}$$

a) A unique invariant distribution φ *for* $\mathbf{P}$ *exists (that is, we have positive recurrence).*

b) $f(z) = \sum_{k=0}^{\infty} p_k z^k - z^d$ *has exactly* d *(not necessarily distinct) roots within* $U_1(0)$*.*

c) Let $\mu_0 + \ldots + \mu_d z^d$ *be a minimal polynomial for these roots, and let* $\varphi = (\varphi_n)_{n\in\mathbb{N}_0}$ *be the invariant distribution with* $\varphi_n \in \mathbb{R}^d$*. Then we have*

$$\varphi_n = \varphi_0 R^n,$$

with

$$R = -\begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} \begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix}^{-1}.$$

d) We have $R = \lim_{N\to\infty} B_N^{-1} B_{N-1}$*, where*

$$B_0 = G_0^{-1}, \qquad 0 = B_{n-1} - \sum_{m=0}^{n} G_{n-m} B_m, \qquad n \in \mathbb{N}.$$

e) Additionally, let

$$F_n = \begin{pmatrix} \sum\limits_{k\geq(n+1)d} p_k & p_{(n+1)d-1} & \cdots & p_{nd+1} \\ \sum\limits_{k\geq(n+1)d+1} p_k & p_{(n+1)d} & \cdots & p_{nd+2} \\ \vdots & \vdots & & \vdots \\ \sum\limits_{k\geq(n+2)d-1} p_k & p_{(n+2)d-2} & \cdots & p_{(n+1)d} \end{pmatrix}.$$

Then $\varphi_0 = c(1\ 0\ \ldots\ 0)R$ *where* $c = -\frac{\sum_{j=0}^{d} \mu_j}{\mu_0}$.

Proof. First, we refer to [Neu81] for some basic statements:

- We have $\varphi_n = \varphi_0 R^n$ where R is the minimal non-negative matrix satisfying

$$\sum_{n=0}^{\infty} R^n G_n = R \qquad \text{and} \qquad R = \left(I - \sum_{n=1}^{\infty} G_n R^{n-1} \right)^{-1} G_0.$$

- φ_0 is characterized by $\varphi_0 \sum_{n=0}^{\infty} R^n F_n = \varphi_0$ and $\varphi_0 (I-R)^{-1} e = 1$ where $e = (1, \ldots, 1)^T \in \mathbb{R}^d$.

- In particular, the matrix–geometric series $\sum_{n=0}^{\infty} R^n$ converges.

- Furthermore, $\sum_{n=0}^{\infty} R^n F_n$ is a stochastic and irreducible (finite) matrix.

Now we prove our statements, throughout the proof let $\mathcal{R} = \mathbb{C}^{d \times d}$, and let $I \in \mathcal{R}$ be the identity matrix.

a) Positive recurrence can be proved by means of drift criteria (see Theorem C.2.6): With state space $\mathbb{N}_0$, we can interprete $\mathbf{P} = (p_{ij})_{i,j \in \mathbb{N}_0}$ as a scalar transition probability matrix $\mathbf{R} = (r_{ij})_{i,j \in \mathbb{N}_0}$ with $r_{i,i+d-j} = p_j$ for $j = 0, \ldots, i$. Now put $g(i) = i$ for $i \geq d$ and $g(i) = 0$ for $i < d$. Then for $i \geq d$, the drift computes to

$$\begin{aligned} d_g(i) &= \sum_{j=0}^{i+d} r_{i,i+d-j} g(i+d-j) - g(i) = \sum_{j=0}^{i} p_j (i+d-j) - i \\ &\leq d - \sum_{j=0}^{i} p_j \cdot j, \end{aligned}$$

and hence $\limsup_{i \to \infty} d_g(i) < 0$, and we can determine a finite set $C \subset \mathbb{N}_0$ and some $\epsilon > 0$ such that $d_g(i) \leq -\epsilon$ for $i \notin C$.

b) We have $f(1) = 0$ and $f'(1) > 0$. Hence, $f(1-\epsilon) < 0$ for some sufficiently small $\epsilon > 0$. With $g(z) = z^d$ and $h(z) = \sum_{k=0}^{\infty} p_k z^k$, we have $f = h - g$, and for $|z| = 1 - \epsilon$, we have

$$|h(z)| \leq h(1-\epsilon) = g(1-\epsilon) + f(1-\epsilon) < g(1-\epsilon) = |g(z)|,$$

and Rouché's Theorem guarantees that g and f both have d roots within $U_{1-\epsilon}(0)$. For $\epsilon \to 0$, we obtain that f has exactly d roots within $U_1(0)$.

c) According to Theorem 9.2.4, the d (not necessarily distinct) minimal roots of f define d linearly independent solutions $\psi^{(1)}, \ldots, \psi^{(d)}$ of the sum equation

$$\pi_{m+d} - \sum_{n=0}^{\infty} \pi_{m+n} p_n = 0, \qquad m \in \mathbb{N}_0, \tag{9.3.1}$$

and we have $\sum\limits_{n=0}^{\infty} \psi_n^{(i)} < \infty$ for $i = 1, \ldots, d$. Now consider roots of f with absolute value 1 additionally; one such root is given by 1. These roots define solutions ψ for (9.3.1), too, but $\sum \psi_n$ will diverge. Hence, Theorem 9.2.4 guarantees that there are exactly d linearly indepent solutions $\psi^{(1)}, \ldots, \psi^{(d)}$ for (9.3.1) with $\sum \psi_n^{(i)} < \infty$, and these solutions are generated by the d roots of f within $U_1(0)$. In particular, we have

$$\left(\psi_0^{(i)}, \ldots, \psi_{d-1}^{(i)}\right) \begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} = - \left(\psi_d^{(i)}, \ldots, \psi_{2d-1}^{(i)}\right) \begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix},$$

where $\mu_0 + \ldots + \mu_d z^d$ is a minimal polynomial for these roots.

Due to $G_0 \in \mathcal{R}^*$, we have $R \in \mathcal{R}^*$, and hence, the rows of R are linearly independent. Now let $\pi_0^{(i)}, \ldots \pi_{d-1}^{(i)}$ be the ith row of R, let $\pi_d^{(i)}, \ldots, \pi_{2d-1}^{(i)}$ be the ith row of R^2,.... Then

$$\sum_{n=0}^{\infty} R^{n+1} G_n = R^2$$

yields that $\pi^{(i)}$ satisfies (9.3.1) for $m = 0, \ldots, d-1$,

$$\sum_{n=0}^{\infty} R^{n+2} G_n = R^3$$

yields that $\pi^{(i)}$ satisfies (9.3.1) for $m = d, \ldots, 2d-1$, and so on. Generally, for $i = 1, \ldots, d$, $\pi^{(i)}$ satisfies (9.3.1) for all $m \in \mathbb{N}_0$. Furthermore, convergence of $\sum R^n$ implies $\sum\limits_{n=0}^{\infty} \pi_n^{(i)} < \infty$ for $i = 1, \ldots, d$. Since the $\pi^{(i)}$ and the $\psi^{(i)}$ are linearly independent, we have

$$\psi^{(i)} = \sum_{j=1}^{d} c_{ij} \pi^{(j)}$$

for all $i = 1, \ldots, d$, and $C = (c_{ij})_{j=1}^{d} \in \mathcal{R}^*$. Hence, we obtain

$$\begin{aligned}
& CR^2 \begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix} \\
= \; & C \begin{pmatrix} \pi_d^{(1)} & \cdots & \pi_{2d-1}^{(1)} \\ \vdots & & \vdots \\ \pi_d^{(d)} & \cdots & \pi_{2d-1}^{(d)} \end{pmatrix} \begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix} \\
= \; & \begin{pmatrix} \psi_d^{(1)} & \cdots & \psi_{2d-1}^{(1)} \\ \vdots & & \vdots \\ \psi_d^{(d)} & \cdots & \psi_{2d-1}^{(d)} \end{pmatrix} \begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix} \\
= \; & - \begin{pmatrix} \psi_0^{(1)} & \cdots & \psi_{d-1}^{(1)} \\ \vdots & & \vdots \\ \psi_0^{(d)} & \cdots & \psi_{d-1}^{(d)} \end{pmatrix} \begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} \\
= \; & -C \begin{pmatrix} \pi_0^{(1)} & \cdots & \pi_{d-1}^{(1)} \\ \vdots & & \vdots \\ \pi_0^{(d)} & \cdots & \pi_{d-1}^{(d)} \end{pmatrix} \begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} \\
= \; & -CR \begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix}.
\end{aligned}$$

Due to $C \in \mathcal{R}^*$ and $R \in \mathcal{R}^*$, we finally obtain

$$R = - \begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} \begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix}^{-1}.$$

d) By transposing the definition of pig (α), we obtain that $\lim_{N\to\infty} B_N^{-1} B_{N-1}$ solves $\sum_{n=0}^{\infty} \alpha_n Z^n = 0$, where

$$\alpha_0 B_0 + I = 0, \qquad \sum_{m=0}^{n} \alpha_{n-m} B_m = 0.$$

Here, we set $\alpha_n = \delta_{n1} I - G_n$, and obtain $-\alpha_0, I - \alpha_1, -\alpha_2, -\alpha_3, \ldots \geq 0$. Due to the Markov-chain interpretation of $\mathbf{P}$, the conditions of Theorem 4.8.1 are met, that is, $\lim_{N\to\infty} B_N^{-1} B_{N-1}$ characterizes the minimal non-negative root, and according to Neuts, this minimal root is R.

e) Up to a constant multiple, φ_0 is uniquely characterized by

$$\varphi_0 S - \varphi_0 = 0 \qquad \text{where} \qquad S = (s_{ij})_{i,j=1}^{d} = \sum_{n=0}^{\infty} R^n F_n.$$

Let $\varphi_0 = (\varphi_{0,i})_{i=1}^{d}$. If

$$\sum_{i=1}^{d} \varphi_{0,i} s_{ij} = \varphi_{0,j}, \qquad j = 2, \ldots, d,$$

we have $\sum \varphi_{0,i} s_{i1} = \varphi_{0,1}$, since otherwise

$$\sum_{j=1}^{d} \varphi_{0,j} \neq \sum_{j=1}^{d} \sum_{i=1}^{d} \varphi_{0,i} s_{ij} = \sum_{i=1}^{d} \varphi_{0,i}.$$

Therefore, we prove that $\varphi_0 = c(1, 0, \ldots, 0)R$ solves $\varphi_0 S = \varphi_0$ up to the first scalar equation. Thus, we may change the first column of S arbitrarily, and in particular, we may replace F_n by G_{n+1}. Up to the first column, we have

$$\begin{aligned} & c(1,0,\ldots,0)R \sum_{n=0}^{\infty} R^n G_{n+1} - c(1,0,\ldots,0)R \\ = \;& c(1,0,\ldots,0) \left(\sum_{n=0}^{\infty} R^{n+1} G_{n+1} - R \right) \\ = \;& -c(1,0,\ldots,0)G_0 = 0. \end{aligned}$$

It remains to prove that $\varphi_0 (I - R)^{-1} e = 1$ yields $c = -\frac{\sum_{j=0}^{d} \mu_j}{\mu_0}$. From the representation of R and $\varphi_0 = c(1, 0, \ldots, 0)R$, we obtain

$$\begin{aligned} 1 \;=\; & (c, 0, \ldots, 0) R (I - R)^{-1} \cdot e \\ =\; & -(c, 0, \ldots, 0) \begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} \\ & \cdot \left(\left(\begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix} + \begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} \right) \right)^{-1} \cdot e \\ =\; & -(c\mu_0, 0, \ldots, 0) \begin{pmatrix} \mu_0 + \mu_d & \mu_{d-1} & \mu_{d-2} & \cdots & \mu_1 \\ \mu_1 & \mu_0 + \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ \mu_2 & \mu_1 & \mu_0 + \mu_d & \vdots & \mu_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mu_{d-1} & \mu_{d-2} & \mu_{d-3} & \cdots & \mu_0 + \mu_d \end{pmatrix}^{-1} \cdot e. \end{aligned}$$

Let $x = (x_1 \ \dots \ x_d)$ be the first row of the inverse. Then

$$\delta_{j1} = \sum_{i=j}^{d} x_i \mu_{i-j} + \sum_{i=1}^{j} x_i \mu_{d+i-j},$$

and hence

$$\begin{aligned} 1 &= \sum_{j=1}^{d} \delta_{j1} = \sum_{j=1}^{d} \sum_{i=j}^{d} x_i \mu_{i-j} + \sum_{j=1}^{d} \sum_{i=1}^{j} x_i \mu_{d+i-j} \\ &= \sum_{i=1}^{d} x_i \left(\sum_{j=1}^{i} \mu_{i-j} + \sum_{j=i}^{d} \mu_{d+i-j} \right) \\ &= \sum_{i=1}^{d} x_i \left(\sum_{j=0}^{d} \mu_j \right), \end{aligned}$$

that is, $xe = \frac{1}{\sum_{j=0}^{d} \mu_j}$. In total, we obtain

$$1 = -c \frac{\mu_0}{\sum_{j=0}^{d} \mu_j}, \qquad \text{and hence} \qquad c = -\frac{\sum_{j=0}^{d} \mu_j}{\mu_0}.$$

□

9.4 Application: Waiting time distribution for $M/D/1$

We consider a problem similar to that in Example 1.3.1: Let W_n be the waiting time of the nth customer in the $G/D/1$ queue with iid interarrival times $T_1, T_2, \dots$ with distribution function F_T and $\mathbb{E}[T_1] = \frac{1}{\lambda} \in (0, \infty)$ and deterministic service times $S_1 = S_2 = \dots = 1$. According to *Lindley's recursion*, we have

$$W_{n+1} = \max\{W_n + 1 - T_{n+1}, 0\}, \qquad n \in \mathbb{N}_0.$$

$(W_n)_{n \in \mathbb{N}_0}$ is a Markov chain with values in $[0, \infty)$, and there is an irreducibility measure (for example, the Dirac measure in 0). For $\lambda < 1$, it is positive recurrent, as can be seen by using drift criteria, see [MT93, Proposition 11.4.1]. As an approximation, we define a Markov chain with values in the discrete state space $\left\{0, \frac{1}{d}, \frac{2}{d}, \dots\right\}$ with $d > 0$ by

$$\tilde{W}_{n+1} = \max\left\{\tilde{W}_n + 1 - \tilde{T}_{n+1}, 0\right\}, \qquad n \in \mathbb{N}_0$$

where

$$\mathbb{P}\left(\tilde{T}_n = \frac{k}{d}\right) = \mathbb{P}\left(T_n \in \left[\frac{k}{d}, \frac{k+1}{d}\right)\right) =: p_k, \qquad k \in \mathbb{N}_0.$$

For $i \in \mathbb{N}_0$ and $j = 1, \ldots, i+1$, we have

$$\mathbb{P}\left(\tilde{W}_1 = \frac{j}{d} \middle| \tilde{W}_0 = \frac{i}{d}\right) = \mathbb{P}\left(\tilde{T}_1 = \frac{i+1-j}{d}\right) = p_{i+1-j},$$

that is, the transition probability matrix has the form required in Theorem 9.3.1, and for $d \to \infty$,

$$\frac{1}{d}\sum_{k=0}^{\infty} k p_k = \sum_{k=0}^{\infty} \frac{k}{d}\mathbb{P}\left(\tilde{T}_n = \frac{k}{d}\right) = \mathbb{E}\left[\tilde{T}_1\right] \to \mathbb{E}\left[T_1\right] = \frac{1}{\lambda} > 1,$$

that is, $\sum\limits_{k=0}^{\infty} k p_k > d$ for sufficiently large d. Hence, the conditions of Theorem 9.3.1 are met, and we have

$$\varphi_n = \left(\pi_n, \pi_{n+\frac{1}{d}}, \ldots, \pi_{n+\frac{d-1}{d}}\right) = -\frac{\sum\limits_{j=0}^{d} \mu_j}{\mu_0}(1, 0, \ldots, 0)R^{n+1},$$

where

$$R = -\begin{pmatrix} \mu_0 & & & \\ \mu_1 & \mu_0 & & \\ \vdots & & \ddots & \\ \mu_{d-1} & \cdots & \cdots & \mu_0 \end{pmatrix} \begin{pmatrix} \mu_d & \mu_{d-1} & \cdots & \mu_1 \\ & \mu_d & & \mu_2 \\ & & \ddots & \vdots \\ & & & \mu_d \end{pmatrix}^{-1},$$

and $\mu_0 + \ldots + \mu_d z^d$ is a minimal polynomial for the roots of $\sum p_k z^k - z^d$ which are located within $U_1(0)$. For concrete distribution functions for T_1, we may use this general setting for determining the invariant distribution π for $\left(\tilde{W}_n\right)_{n\geq 0}$, and by taking $d \to 0$, we may obtain the invariant distribution for $(W_n)_{n\geq 0}$.

Here, we consider the case of $M/D/1$, that is, the interarrival times $T_1, T_2, \ldots$ are exponentially distributed with parameter λ, that is

$$\mathbb{P}(T_n \leq t) = 1 - e^{-\lambda t}, \qquad t \geq 0.$$

Due to continuity, with $q = e^{-\frac{\lambda}{d}}$, we obtain

$$p_k = \mathbb{P}\left(T_n \geq \frac{k}{d}\right) - \mathbb{P}\left(T_n \geq \frac{k+1}{d}\right) = q^k - q^{k+1} = (1-q)q^k, \qquad k \in \mathbb{N}_0.$$

For obtaining the coefficients of the minimal polynomial of the roots, we write

$$\begin{aligned} \sum_{k=0}^{\infty} p_k z^k - z^d &= \frac{(1-qz)z^d - (1-q)}{1-qz} = \frac{q\left(1-z^{d+1}\right) - (1-z^d)}{1-qz} \\ &= -\frac{1-z}{1-qz}\left(q\sum_{k=0}^{d} z^k - \sum_{k=0}^{d-1} z^k\right) = -\frac{1-z}{1-qz}\left(qz^d - \sum_{k=0}^{d-1}(1-q)z^k\right), \end{aligned}$$

and obtain $\mu_0 = \ldots = \mu_{d-1} = -(1-q)$ and $\mu_d = q$. This results in

$$\begin{aligned}
R &= \begin{pmatrix} 1-q & & & \\ 1-q & 1-q & & \\ \vdots & & \ddots & \\ 1-q & 1-q & \ldots & 1-q \end{pmatrix} \begin{pmatrix} q & -(1-q) & \ldots & -(1-q) \\ & q & \ldots & -(1-q) \\ & & \ddots & \vdots \\ & & & q \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1-q & & & \\ 1-q & 1-q & & \\ \vdots & & \ddots & \\ 1-q & 1-q & \ldots & 1-q \end{pmatrix} \left(I - \begin{pmatrix} 1-q & 1-q & \ldots & 1-q \\ & 1-q & \ldots & 1-q \\ & & \ddots & \vdots \\ & & & 1-q \end{pmatrix} \right)^{-1} \\
&= \begin{pmatrix} 1-q & & & \\ 1-q & 1-q & & \\ \vdots & & \ddots & \\ 1-q & 1-q & \ldots & 1-q \end{pmatrix} \begin{pmatrix} \frac{1}{q} & \frac{1-q}{q^2} & \frac{1-q}{q^3} & \ldots & \frac{1-q}{q^d} \\ & \frac{1}{q} & \frac{1-q}{q^2} & \ldots & \frac{1-q}{q^{d-1}} \\ & & \ddots & \ddots & \vdots \\ & & & \frac{1}{q} & \frac{1-q}{q^2} \\ & & & & \frac{1}{q} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-q}{q} & \frac{(1-q)(1-q)}{q^2} & \frac{(1-q)(1-q)}{q^3} & \ldots & \frac{(1-q)(1-q)}{q^d} \\ \frac{1-q}{q} & \frac{1-q}{q^2} & \frac{(1-q)(1-q^2)}{q^3} & \ldots & \frac{(1-q)(1-q^2)}{q^d} \\ \frac{1-q}{q} & \frac{1-q}{q^2} & \frac{1-q}{q^3} & \ldots & \frac{(1-q)(1-q^3)}{q^d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1-q}{q} & \frac{1-q}{q^2} & \frac{1-q}{q^3} & \ldots & \frac{1-q}{q^d} \end{pmatrix},
\end{aligned}$$

that is,

$$\begin{aligned}
r_{i0} &= \frac{1-q}{q}, \qquad i = 0, \ldots, d-1, \\
r_{ij} &= \frac{(1-q)(1-q^{i+1})}{q^{j+1}} = \frac{1-q}{q^{j+1}} - \frac{1-q}{q^{j-i}}, \qquad j = 1, \ldots, d-1,\ i = 0, \ldots, j-1, \\
r_{ij} &= \frac{1-q}{q^{j+1}}, \qquad j = 1, \ldots, d-1,\ i = j, \ldots, d-1.
\end{aligned}$$

For φ_0, we obtain

$$\begin{aligned}
\varphi_0 &= -\frac{\sum_{j=0}^{d} \mu_j}{\mu_0}(1, 0, \ldots, 0)R = \frac{q - d(1-q)}{1-q}(1, 0, \ldots, 0)R \\
&= \left(1 - d\frac{1-q}{q}\right)\left(1, \frac{1-q}{q}, \frac{1-q}{q^2}, \ldots, \frac{1-q}{q^{d-1}}\right).
\end{aligned}$$

We have $\pi_0 = 1 - d\frac{1-q}{q}$, and for $n \in \mathbb{N}_0$ and $k = 1, \ldots, d$, we prove

$$\pi_{n+\frac{k}{d}} = \left(1 - d\frac{1-q}{q}\right) \sum_{m=0}^{n} \frac{(q-1)^m}{q^{(n-m)d+k}} \left(\binom{(n-m)d+k}{m} - q \binom{(n-m)d+k-1}{m} \right). \tag{9.4.1}$$

by induction. In the proof, we make use of telescope sums extensively:

- For $n=0$ and $k=1,\ldots,d-1$, $\pi_{0+\frac{k}{d}}$ is an entry of φ_0, we have $\pi_{0+\frac{k}{d}} = \frac{1-q}{q^k} = \frac{\binom{k}{0}-q\binom{k-1}{0}}{q^k}$.
- For $n=0$ and $k=d$, we have

$$\pi_{0+\frac{d}{d}} = \pi_1 = \pi_0 r_{00} + \sum_{j=1}^{d-1} \pi_{\frac{j}{d}} r_{j0} = \pi_0 \frac{1-q}{q}\left(1 + \sum_{j=1}^{d-1} \frac{1-q}{q^j}\right) = \pi_0 \frac{1-q}{q}\left(1 + \frac{1}{q^{d-1}} - 1\right) = \pi_0 \cdot \frac{1-q}{q^d}.$$

- For $n=1$ and $k=1,\ldots,d-1$, we have

$$\begin{aligned}
\pi_{1+\frac{k}{d}} &= \pi_0 r_{0k} + \sum_{j=1}^{d-1} \pi_{\frac{j}{d}} r_{jk} \\
&= \pi_0 \left(\frac{1-q}{q^{k+1}} - \frac{1-q}{q^k} + \frac{1-q}{q^{k+1}} \sum_{j=1}^{d-1} \cdot \frac{1-q}{q^j} - \sum_{j=1}^{k-1} \frac{1-q}{q^{k-j}} \cdot \frac{1-q}{q^j} \right) \\
&= \pi_0 \left(\frac{1-q}{q^{k+1}} - \frac{1-q}{q^k} + \frac{1-q}{q^{k+1}} \left(\frac{1}{q^{d-1}} - 1 \right) - (k-1)\frac{(1-q)^2}{q^k} \right) \\
&= \pi_0 \left(\frac{1-q}{q^{d+k}} - \frac{1-q+(k-1)(1-q)^2}{q^k} \right) \\
&= \pi_0 \left(\frac{1-q}{q^{d+k}} + \frac{(q-1)\left(k+q(k-1)\right)}{q^k} \right).
\end{aligned}$$

- For $n \geq 2$, we have

$$\begin{aligned}
\pi_{n-1+\frac{d}{d}} &= \pi_n = \pi_{n-1} r_{00} + \sum_{j=1}^{d-1} \pi_{n-1+\frac{j}{d}} r_{j0} \\
&= \pi_0 \frac{1-q}{q} \left(\sum_{m=0}^{n-2} \frac{(q-1)^m}{q^{(n-2-m)d+d}} \left(\binom{(n-2-m)d+d}{m} - q\binom{(n-2-m)d+d-1}{m} \right) \right. \\
&\qquad \left. + \sum_{j=1}^{d-1} \sum_{m=0}^{n-1} \frac{(q-1)^m}{q^{(n-1-m)d+j}} \left(\binom{(n-1-m)d+j}{m} - q\binom{(n-1-m)d+j-1}{m} \right) \right) \\
&= \pi_0 \left(\frac{1-q}{q} \sum_{m=0}^{n-2} \frac{(q-1)^m}{q^{(n-m-1)d}} \binom{(n-m-1)d}{m} + \sum_{m=0}^{n-2} \frac{(q-1)^{m+1}}{q^{(n-m-1)d}} \binom{(n-m-1)d-1}{m} \right. \\
&\qquad \left. + \frac{1-q}{q} \sum_{m=0}^{n-1} (q-1)^m \left(\frac{\binom{(n-m)d-1}{m}}{q^{(n-m)d-1}} - \frac{\binom{(n-m-1)d}{m}}{q^{(n-m-1)d}} \right) \right) \\
&= \pi_0 \left(\frac{1-q}{q} \sum_{m=0}^{n-2} \frac{(q-1)^m}{q^{(n-m-1)d}} \binom{(n-m-1)d}{m} + \sum_{m=1}^{n-1} \frac{(q-1)^m}{q^{(n-m)d}} \binom{(n-m)d-1}{m-1} \right. \\
&\qquad \left. + \frac{1-q}{q^m} + \left(\frac{1}{q} - 1\right) \sum_{m=1}^{n-1} (q-1)^m \frac{\binom{(n-m)d-1}{m}}{q^{(n-m)d-1}} - \frac{1-q}{q} \sum_{m=0}^{n-2} (q-1)^m \frac{\binom{(n-m-1)d}{m}}{q^{(n-m-1)d}} \right) \\
&= \pi_0 \left(\frac{1-q}{q^m} + \sum_{m=1}^{n-1} \frac{(q-1)^m}{q^{(n-m)d}} \left(\binom{(n-m)d-1}{m-1} + \binom{(n-m)d-1}{m} \right) \right. \\
&\qquad \left. - \sum_{m=1}^{n-1} \frac{(q-1)^m}{q^{(n-m)d-1}} \binom{(n-m)d-1}{m} \right) \\
&= \pi_0 \sum_{m=0}^{n-1} \frac{(q-1)^m}{q^{(n-m)d}} \left(\binom{(n-m)d}{m} - q\binom{(n-m)d-1}{m} \right).
\end{aligned}$$

- For $n \geq 2$ and $k=1,\ldots,d-1$, we additionally make use of the popular combinatorial formula

$$\sum_{i=m-1}^{M-1} \binom{i}{m-1} = \binom{M}{m},$$

which can be proved directly by combinatorial reasons or by a simple induction. By building differences, we obtain the formula

$$\sum_{i=M_0}^{M_1-1}\binom{i}{m-1}=\binom{M_1}{m}-\binom{M_0}{m}.$$

Using the structure of the entries of R, we derive

$$\begin{aligned}
\pi_{n+\frac{k}{d}} &= \sum_{j=0}^{d-1}\pi_{n-1+\frac{j}{d}}r_{jk}=\frac{1}{q^k}\cdot\frac{1-q}{q}\sum_{j=0}^{d-1}\pi_{n-1+\frac{j}{d}}-\sum_{j=0}^{k-1}\frac{1-q}{q^{k-j}}\pi_{n-1+\frac{j}{d}}\\
&= \frac{\pi_n}{q^k}-\frac{1-q}{q^k}\pi_{n-1}-\sum_{j=1}^{k-1}\frac{1-q}{q^{k-j}}\pi_{n-1+\frac{j}{d}}\\
&= \pi_0\Bigg(\sum_{m=0}^{n-1}\frac{(q-1)^m}{q^{(n-m)d+k}}\left(\binom{(n-m)d}{m}-q\binom{(n-m)d-1}{m}\right)\\
&\qquad+\sum_{m=0}^{n-2}\frac{(q-1)^{m+1}}{q^{(n-1-m)d+k}}\left(\binom{(n-1-m)d}{m}-q\binom{(n-1-m)d-1}{m}\right)\\
&\qquad+\sum_{j=1}^{k-1}\sum_{m=0}^{n-1}\frac{(q-1)^{m+1}}{q^{(n-1-m)d+k}}\left(\binom{(n-1-m)d+j}{m}-q\binom{(n-1-m)d+j-1}{m}\right)\Bigg)\\
&= \pi_0\Bigg(\frac{1-q}{q^{nd+k}}+\sum_{m=1}^{n-1}\frac{(q-1)^m}{q^{(n-m)d+k}}\left(\binom{(n-m)d}{m}-q\binom{(n-m)d-1}{m}\right)\\
&\qquad+\sum_{m=1}^{n-1}\frac{(q-1)^m}{q^{(n-m)d+k}}\left(\binom{(n-m)d}{m-1}-q\binom{(n-m)d-1}{m-1}\right)\\
&\qquad+\sum_{m=1}^{n}\sum_{j=1}^{k-1}\frac{(q-1)^m}{q^{(n-m)d+k}}\left(\binom{(n-m)d+j}{m-1}-q\binom{(n-m)d+j-1}{m-1}\right)\Bigg)\\
&= \pi_0\Bigg(\frac{1-q}{q^{nd+k}}+\sum_{m=1}^{n-1}\frac{(q-1)^m}{q^{(n-m)d+k}}\left(\binom{(n-m)d+1}{m}-q\binom{(n-m)d}{m}\right)\\
&\qquad+\sum_{m=1}^{n}\frac{(q-1)^m}{q^{(n-m)d+k}}\left(\binom{(n-m)d+k}{m}-\binom{(n-m)d+1}{m}\right)\\
&\qquad-\sum_{m=1}^{n}\frac{(q-1)^m}{q^{(n-m)d+k}}\cdot q\left(\binom{(n-m)d+k-1}{m}-\binom{(n-m)d}{m}\right)\Bigg)\\
&= \pi_0\sum_{m=0}^{n}\frac{(q-1)^m}{q^{(n-m)d+k}}\left(\binom{(n-m)d+k}{m}-q\binom{(n-m)d+k-1}{m}\right)
\end{aligned}$$

where in the last step we used $\binom{0}{n}=\binom{1}{n}=0$ for $n\geq 2$.

By writing $i=nd+k$ with $n=\left\lfloor\frac{i}{d}\right\rfloor$ in (9.4.1), we obtain

$$\begin{aligned}
\pi_{\frac{i}{d}} &= \left(1-d\frac{1-q}{q}\right)\sum_{m=0}^{\left\lfloor\frac{i}{d}\right\rfloor}\frac{(q-1)^m}{q^{i-md}}\left(\binom{i-md}{m}-q\binom{i-1-md}{m}\right)\\
&= \left(1-d\frac{1-q}{q}\right)\sum_{m=0}^{\infty}\frac{(q-1)^m}{q^{i-md}}\left(\binom{i-md}{m}-q\binom{i-1-md}{m}\right),
\end{aligned}$$

and we derive the distribution function

$$\begin{aligned}
\mathbb{P}\left(\tilde{W}_\infty \le \frac{k}{d}\right) &= \sum_{i=0}^{k} \pi_{\frac{i}{d}} = \pi_0 \left(1 + \sum_{i=1}^{k} \sum_{m=0}^{\infty} \frac{(q-1)^m}{q^{i-md}} \left(\binom{i-md}{m} - q\binom{i-1-md}{m}\right)\right) \\
&= \pi_0 \left(1 + \sum_{m=0}^{\infty} (q-1)^m \left(\frac{\binom{k-md}{m}}{q^{k-md}} - \frac{\binom{0-md}{m}}{q^{-md}}\right)\right) \\
&= \pi_0 \sum_{m=0}^{\infty} \frac{(q-1)^m}{q^{k-md}} \binom{k-md}{m} \\
&= \left(1 - \frac{d(1-q)}{q}\right) \sum_{m=0}^{\lfloor \frac{k}{d} \rfloor} \frac{(q-1)^m}{q^{k-md}} \binom{k-md}{m},
\end{aligned}$$

that is

$$\begin{aligned}
\mathbb{P}\left(\tilde{W}_\infty \le t\right) &= \mathbb{P}\left(\tilde{W}_\infty \le \frac{\lfloor td \rfloor}{d}\right) = \left(1 - \frac{d(1-q)}{q}\right) \sum_{m=0}^{\lfloor \frac{\lfloor td \rfloor}{d} \rfloor} \frac{(q-1)^m}{q^{\lfloor td \rfloor - md}} \binom{\lfloor td \rfloor - md}{m} \\
&= \left(1 - \frac{d(1-q)}{q}\right) \sum_{m=0}^{\lfloor \frac{\lfloor td \rfloor}{d} \rfloor} \frac{(d(q-1))^m}{q^{\lfloor td \rfloor - md}} \cdot \frac{\binom{\lfloor td \rfloor - md}{m}}{d^m}.
\end{aligned}$$

By writing $\frac{\binom{\lfloor td \rfloor - md}{m}}{d^m} = \frac{1}{m!}\left(\frac{\lfloor td \rfloor}{d} - m\right)\left(\frac{\lfloor td \rfloor - 1}{d} - m\right) \ldots \left(\frac{\lfloor td \rfloor - m + 1}{d} - m\right)$, and using $d(1-q) = \frac{1 - e^{-\frac{\lambda}{d}}}{\frac{1}{d}} \to \lambda$, for $d \to \infty$, we obtain

$$F(x) := \mathbb{P}(W_\infty \le t) = (1-\lambda) \sum_{m=0}^{\lfloor t \rfloor} \frac{(-\lambda)^m}{e^{-\lambda(t-m)}} \cdot \frac{(t-m)^m}{m!} = (1-\lambda) \sum_{m=0}^{\lfloor t \rfloor} \frac{(-\lambda(t-m))^m}{m!} e^{\lambda(t-m)}.$$

The waiting time distribution for the $M/D/1$-queue is quite well–known, see [Cro34; Fra01]. As pointed out above, our method is not restricted to the $M/D/1$-queue, and hence, it might be helpful in situations in which no exact invariant distribution for the waiting time is known.

9.5 Literature review

Characterizing roots of quadratic polynomials by periodic continued fractions has a very long tradition, see [Per54] for more details. In fact, periodic continued fractions are the oldest and most famous ones, e.g.

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}} = \frac{1 + \sqrt{5}}{2}$$

is the golden ratio.

Periodic continued fractions in Banach algebras have also been used in the literature, see [BF94; CH90] for applications to Ricatti equations.

Implicitely, Neuts [Neu81] introduced periodic gcfs when discussing matrix-geometric methods for Markov chains. As demonstrated in section 9.3, in a certain sense, the matrix R defined by Neuts is a periodic gcf. In the Markov chain context, convergence to a minimal non-negative solution is guaranteed, and hence, a large part of the discussion of periodic gcfs in Banach algebras is inspired by Neuts work.

Chapter 10

Further research and open problems

10.1 Minimal roots

In Theorem 9.3.1, we proved that a (transposed) matrix-valued periodic gcf is capable of characterizing the d roots of a certain analytic function which are located within $U_1(0)$. Here, the question arises whether or not matrix-valued gcfs are capable of characterizing the d minimal roots of meromorphic functions in more general settings.

In Theorem 9.2.3, we gave a sufficient criterion for a analytic function to have no roots in a half-plain, but unfortunately, this condition does not apply to 'interesting' functions. Hence, a more elegant way to determine all roots or to determine regions without roots would be desirable.

10.2 Poincaré-type equations

Roots of holomorphic functions are quite closely related to solutions for sum equations with constant coefficients, see [Per21]. In case that the coefficients are non-constant but convergent, we obtain *Poincaré–type* equation. For Poincaré–type difference equations in $\mathbb{C}$, literature concerning solutions generated by continued fractions can be found [Per21; Han91]. Hence, a natural task is considering gcfs for which the coefficients are non-periodic, but converge.

10.3 Analyticity

Let $\mathbf{H}(z)$ depend on $z \in G \subset \mathbb{C}$, and let $\mathbf{H}(z)$ meet some Pringsheim-type condition for all $z \in G$. Then an interesting question concerns the analytic properties of $z \mapsto \operatorname{gcf}(\mathbf{H}(z))$.

Denk and Riederle [DR82] proved that their Pringsheim-type condition implies that $z \mapsto \operatorname{gcf}(\mathbf{H}(z))$ is holomorphic.

If $\mathbf{H}(z)$ satisfies the conditions of 3.2.5 for all $z \in G$ with some region G and if the entries of $\mathbf{H}(z)$ are holomorphic within G, $||S(\mathbf{P}(z), 0, 0, \mathbb{N}_{1,N})||$ is holomorphic and uniformly bounded

(by 1). Hence, it seems realistic to prove that

$$z \mapsto \operatorname{gcf}(\mathbf{H}(z)) = I - S(\mathbf{P}(z), 0, 0, \mathbb{N})$$

is holomorophic by means of Montel's Theorem.

If $\mathbf{H}(z)$ satisfies the conditions of some Pringsheim–type criterion for all $z \in B$ where B is compact, a natural question concerns continuity of $z \mapsto \mathbf{H}(z)$. However, proving analytic properties of $z \mapsto \mathbf{H}(z)$ requires further research.

10.4 Application to differential equations

In [Per09], Perron considered the differential equation $y = (1+x)y'' + x^2 y'''$, he put $y(x) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} x^n$, and obtained the difference equation $\gamma_n = n^2 \gamma_{n+1} + \gamma_{n+2}$.
Hence, the problem of solving the differential equation is transformed into solving the difference equation. We did not consider the application of our theory to differential equations at all, and thus, this application is a direction of further research. In particular, in stochastics, results could be applied to continuous-time Markov processes, e.g. diffusion processes.

10.5 Transient distributions

For systems of linear differential equations with constant coefficients, Laplace transformation yields systems of linear equations. In particular, the Kolmogorov forward equation $\pi'(t) = \pi(t)\mathbf{Q}$ for the transient distributions $\pi(t)$ of a CTMC X with regular generator $\mathbf{Q}$ can be transformed into an (inhomogeneous) system of equations by taking Laplace transforms. Usually, this is avoided in context of numerical methods since

- the inverse Laplace transform causes numerical problems and
- for each $i \in E$, we have to take an inverse Laplace transform for obtaining $\pi_i(t)$.

With ideas similar to those for directly computing total absorption probabilities or ψf, it should be possible to compute $\overline{G}(s)$ directly, where $\overline{G}$ is the Laplace transform of $t \mapsto \mathbb{E}[g(X_t)]$. Hence, we have to take only one inverse Laplace transform, and this might be possible numerically.

10.6 Parameter estimation in Markov models

Some of the algorithms for Markov chains in chapter 6 can be differentiated, that is, if the transition probabilities or rates depend on some parameter ϑ, we are able to compute approximations for $\psi f(\vartheta)$ and $\frac{\partial}{\partial \vartheta} \psi f(\vartheta)$ silmutaneously. In this context, the question arises whether this enables us to estimate the parameter ϑ efficiently.

Appendix A

Convergence of series in Banach spaces and algebras

In this chapter, we briefly state definitions and results concerning convergence, absolute convergence and unconditional convergence in Banach spaces and Banach algebras. For more details, we refer to [Hei11, chapter 3].

A.1 Convergence and unconditional convergence

In a Banach space or algebra $\mathcal{R}$ with norm $||\cdot||$, a series

$$\sum_{i=1}^{\infty} a_i$$

with $a_i \in \mathcal{R}$ is said to be *convergent* with value a if

$$\left|\left| a - \sum_{i=1}^{N} a_i \right|\right|$$

converges to 0 for $N \to \infty$.

If $\sum\limits_{i=1}^{\infty} a_{\sigma(i)}$ converges for any permutation σ of $\mathbb{N}$, $\sum a_i$ is said to be *unconditionally convergent*. It then follows that $a = \sum\limits_{i=1}^{\infty} a_{\sigma(i)}$ does not depend on σ, see [Hei11, chapter 3]. In the situation of unconditional convergence, it makes sense to write $\sum\limits_{i \in I} a_i$ for any countable index set I since the value does not depend on the enumeration of I.

There are many characterizations of unconditional convergence in Banach algebras, in the next results we concentrate on those we need in this work.

Lemma A.1.1. *Let $\mathcal{R}$ be a Banach space or Banach algebra, and let $a_i \in \mathcal{R}$ for all $i \in \mathbb{N}$. Then the following statements are equivalent:*

a) $\sum_{i=1}^{\infty} a_i$ *converges unconditionally.*

b) $\sum_{i=1}^{\infty} w_i a_i$ *converges for any bounded* $\mathbb{C}$*- or* $\mathbb{R}$*-valued sequence* $(w_i)_{i\in\mathbb{N}}$ *of weights.*

c) $\sum_{i=1}^{\infty} a_{i_n}$ *converges for any sequence* $(i_n)_{n\in\mathbb{N}}$ *with* $1 \le i_1 < i_2 < i_3 < \ldots$.

d) For all $\epsilon > 0$*, there is some* $N \in \mathbb{N}$ *such that for all finite* $J \subset \mathbb{N}$ *with* $\min(J) > N$*, we have* $\left|\left|\sum_{i\in J} a_i\right|\right| < \epsilon$.

e) For all $\epsilon > 0$*, there is some* $N \in \mathbb{N}$ *such that for all* $J \subset \mathbb{N}$ *with* $\min(J) > n$*, we have* $\left|\left|\sum_{i\in J} a_i\right|\right| < \epsilon$.

Proof. a) $\Leftrightarrow$ b) $\Leftrightarrow$ c) $\Leftrightarrow$ d) is a standard result, see [Hei11, chapter 3], and e) $\Rightarrow$ d) is trivial. Now let d) hold, and let $\epsilon > 0$. Then there are N_1, N_2, N_3 such that for all finite $K \subset \mathbb{N}$ with $\min(K) > N_k$, we have $\left|\left|\sum_{i\in K} a_i\right|\right| < \frac{\epsilon}{2^k}$. For $J \subset \mathbb{N}$ with $\min(J) > N := N_1$, we write $K_k := J \cap \{N_k + 1, \ldots, N_{k+1}\}$ and obtain

$$\left|\left|\sum_{i\in J} a_i\right|\right| = \left|\left|\sum_{k=1}^{\infty}\sum_{i\in K_k} a_i\right|\right| \le \sum_{k=1}^{\infty}\left|\left|\sum_{i\in K_k} a_i\right|\right| < \sum_{k=1}^{\infty}\frac{\epsilon}{2^k} = \epsilon,$$

that is, we have d) $\Rightarrow$ e). □

Theorem A.1.1. *Let* $\mathcal{R}$ *be a Banach space or Banach algebra, let* $a_i \in \mathcal{R}$ *for all* $i \in I$ *with some countable index set* I*, and let* $\sum_{i\in I} a_i$ *converge unconditionally.*

a) Let $I_n \uparrow I$ *for* $n \to \infty$*, that is, let* $I_1 \subset I_2 \subset I_3 \subset \ldots$ *with* $\bigcup_{n=1}^{\infty} I_n = I$*. Then* $\sum_{i\in I_n} a_i$ *converges unconditionally for all* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty}\sum_{i\in I_n} a_i = \sum_{i\in I} a_i.$$

b) Let $J_1 \dot\cup J_2 \dot\cup \ldots = I$ *be a finite or countably infinite partition of* I*. Then* $\sum_{i\in J_n}$ *converges unconditionally for all* n*, and*

$$\sum_{i\in J_1} a_i + \sum_{i\in J_2} a_i + \ldots = \sum_{i\in I} a_i.$$

Proof. Without restriction, we assume $I = \mathbb{N}$. Furthermore, a) $\Rightarrow$ c) in Lemma A.1.1 yields that every subseries and every subseries of the subseries converges, and c) $\Rightarrow$ a) in Lemma A.1.1 shows that every subseries converges unconditionally. Hence $\sum_{i\in I_n} a_i$ and $\sum_{i\in J_n} a_i$ converge unconditionally. Statement e) in Lemma A.1.1 directly yields a) since $\min(I_n) \ge N$ for all $n \ge n_0$ with some $n_0 \in \mathbb{N}$. b) is obtained by setting $I_n = \bigcup_{m=1}^{n} J_m$ where $J_m = \emptyset$ for large m in case of a finite partition. □

Easy considerations yield

Lemma A.1.2. *Let I and J be countable sets, $a_i, b_j, c \in \mathcal{R}$ for $i \in I$ and $j \in J$, let $\sum_{i\in I} a_i$ and $\sum_{j\in J} b_j$ converge unconditionally. Then*

$$\sum_{i\in I} ca_i = c\sum_{i\in I} a_i, \qquad \sum_{i\in I}(a_i c) = \left(\sum_{i\in I} a_i\right) c, \qquad \sum_{(i,j)\in I\times J} a_i b_j = \sum_{i\in I} a_i \sum_{j\in J} b_j$$

converge unconditionally.

A.2 Absolute convergence

Absolute convergence is a simple criterion for unconditional convergence. Still, we follow [Hei11].

The series $\sum_{i=1}^{\infty} a_i$ is said to be *absolutely convergent* if $\sum_{i=1}^{\infty} ||a_i||$ converges in $\mathbb{R}$. Due to $\left|\left|\sum_{n=N+1}^{\infty} a_n\right|\right| \leq \sum_{n=N+1}^{\infty} ||a_n||$, absolute convergence implies convergence. More than that, absolute convergence implies unconditional convergence.

Theorem A.2.1. *Let $\sum_{i\in I} a_i$ converge absolutely for some countable index set I.*

a) Then $\sum_{i\in I} a_i$ converges unconditionally.

b) For $I_n \uparrow I$, $\sum_{i\in I_n} a_i$ converges absolutely, and we have

$$\lim_{n\to\infty} \sum_{i\in I_n} a_i = \sum_{i\in I} a_i.$$

c) For a partition $I = J_1 \dot\cup J_2 \dot\cup \ldots$, $\sum_{i\in J_n} a_i$ converges absolutely, and we have

$$\sum_{i\in J_1} a_i + \sum_{i\in J_2} a_i + \ldots = \sum_{i\in I} a_i.$$

Proof. a) See e.g. [Hei11], the proof is similar to that of b).

b) Absolute convergence of $\sum_{i\in I_n}$ is obvious. The statement then follows from a) and Theorem A.1.1, but due to its importance, we give a direct proof here.

Let $\epsilon > 0$. Since $\sum_{i\in I} ||a_i|| < \infty$, there is some finite $J \subset I$ such that $\sum_{i\in I\setminus J} ||a_i|| < \epsilon$. Since $I_n \uparrow I$, we find some $n_0 \in \mathbb{N}$ such that $I_n \supset J$ for all $n \geq n_0$, that is, $I \setminus I_n \subset I \setminus J$ for $n \geq n_0$, and hence, for $n \geq n_0$, we have

$$\left|\left|\sum_{i\in I} a_i - \sum_{i\in I_n} a_i\right|\right| = \left|\left|\sum_{i\in I\setminus I_n} a_i\right|\right| \leq \sum_{i\in I\setminus I_n} ||a_i|| \leq \sum_{i\in I\setminus J} ||a_i|| < \epsilon.$$

c) Follows from b) with $I_n = \bigcup_{m=1}^{n} J_m$.

□

In $\mathbb{C}$, the easiest convergent series is the geometric series $\sum q^n$. The name of its generalization in Banach algebras depends on the context it is used in. Sometimes it is called *geometric series* too, especially in matrix algebras it is referred to as *matrix geometric series*, in functional analysis and operator theory it is called *Neumann series*. We will call it *geometric series* and give a basic statement.

Lemma A.2.1. *Let $\sum_{n=0}^{\infty} s^n$ converge for some $s \in \mathcal{R}$. Then*

$$(I-s)^{-1} = \sum_{n=0}^{\infty} s^n$$

exists. If $||s|| < 1$ the series $\sum s^n$ converges absolutely and for every $r \in \mathcal{R}$ we have

$$\big|\big|r \cdot (I-s)^{-1}\big|\big| \leq \frac{||r||}{1-||s||} \qquad \textit{and} \qquad \big|\big|(I-s)^{-1} \cdot r\big|\big| \leq \frac{||r||}{1-||s||}.$$

Proof. The convergence of $\sum_{n=0}^{\infty} s^n$ implies $\lim_{N\to\infty} s^N = 0$. Thus, we have

$$(I-s)\sum_{n=0}^{\infty} s^n = \lim_{N\to\infty}\left(\sum_{n=0}^{N} s^n - \sum_{n=0}^{N} s^{n+1}\right) = \lim_{N\to\infty}\left(I - s^{N+1}\right) = I$$

and

$$\sum_{n=0}^{\infty} s^n(I-s) = \lim_{N\to\infty}\left(\sum_{n=0}^{N} s^n - \sum_{n=0}^{N} s^{n+1}\right) = \lim_{N\to\infty}\left(I - s^{N+1}\right) = I.$$

For $||s|| < 1$, $||s^n|| \leq ||s||^n$ implies the absolute convergence of $\sum s^n$ as well as

$$\big|\big|r \cdot (I-s)^{-1}\big|\big| = \left|\left|\sum_{n=0}^{\infty} r s^n\right|\right| \leq ||r|| \sum_{n=0}^{\infty} ||s||^n = \frac{||r||}{1-||s||}$$

and

$$\big|\big|(I-s)^{-1} \cdot r\big|\big| = \left|\left|\sum_{n=0}^{\infty} s^n r\right|\right| \leq ||r|| \sum_{n=0}^{\infty} ||s||^n = \frac{||r||}{1-||s||}.$$

□

Note that for $||I|| > 1$ (example: Frobenius norm in matrix algebras) we cannot state $\big|\big|(I-s)^{-1}\big|\big| \leq \frac{1}{1-||s||}$ (for example, it would be wrong for $s = 0$). The formulation in Lemma A.2.1 may seem a bit lengthy but the estimates still hold if $||I|| > 1$.

Appendix B

Operator algebras and positive operators

Here, we want to give a brief overview concerning operator algebras, and in particular, positive operators. For details (and for those proofs which we do not state here), we refer to [MN91; Nag86].

B.1 Operator algebras

Let F be a normed vector space, and let $f, g : F \to F$ be linear operators, that is, linear mappings. Then we can define an addition $f+g$ by $(f+g)(a) = f(a)+g(a)$, a multiplication fg by $(fg)(a) := (f \circ g)(a) = f(g(a))$, and the *operator norm*

$$||f|| = \sup_{a \in F \setminus \{0\}} \frac{||f(a)||}{||a||} = \sup_{||a||=1} ||f(a)|| = \sup_{||a|| \leq 1} ||f(a)|| \,.$$

If $||f|| < \infty$, f is said to be *bounded*. An important result is that a linear operator is bounded if and only if it is continuous. Endowed with $+, \cdot, ||\cdot||$,

$$L(F) = \{f : F \to F \text{ linear and bounded}\}$$

becomes a normed algebra. If F is a Banach space (that is, complete), $L(F)$ is a Banach algebra. We define *operator algebras* to be structures of the form $L(F)$ with a Banach space F. Note that in the literature, the definition of an operator algebra sometimes differs:

- Some authors are less restrictive and refer to $L(F)$ as operator algebra for any normed vector space F.
- Some authors are more restrictive and refer to $L(F)$ as operator algebra if F is a separable Hilbert space.

B.2 Positivity on Banach lattices

Positive operators are defined to be linear operators from a Banach lattice mapping positive elements to positive elements. We begin with defining Banach lattices.

Definition B.2.1. A real vector space F is said to be a *vector lattice* if

- there is an order $\leq$ in the sense that
  - $\forall x \in F:\ x \leq x$,
  - $\forall x, y \in F:\ x \leq y, y \leq x \ \Rightarrow\ x = y$,
  - $\forall x, y, z \in F:\ x \leq y, y \leq z \ \Rightarrow\ x \leq z$,
- the order relation is compatible with the vector operations, that is
  - $\forall x, y, z \in F:\ x \leq y \ \Rightarrow x + z \leq y + z$,
  - $\forall \alpha > 0:\ 0 \leq x \ \Rightarrow 0 \leq \alpha x$,
- for any two elements $x, y \in F$, there is a least upper bound $\sup(x, y)$ and a greatest lower bound $\inf(x, y)$.

By $F_+ := \{x:\ 0 \leq x\}$ we denote the *positive cone*, that is the set of all *positive* elements.

If F is a vector lattice, for $x \in F$, we can define

$$|x| = \sup(x, -x) \in F_+.$$

If F is a vector lattice and a Banach space at the same time, and if

$$\forall x, y \in F:\ \ |x| \leq |y| \ \ \Rightarrow\ \ ||x|| \leq ||y||\,, \tag{B.2.1}$$

F is said to be a *real Banach lattice*.

Obviously, $\mathbb{R}^d$ can be interpreted as a Banach lattice by defining $x \leq y$ componentwise and taking an arbitrary p-norm. Similarly, let $(\Omega, \mathcal{A}, \mu)$ be a measure space with a σ-finite measure μ, let $p \in [1, \infty]$ and consider $L^p(\Omega, \mathcal{A}, \mu)$ with the usual p-norm. By defining

$$0 \leq f \ \Leftrightarrow\ \mu\left(\{x \in \Omega:\ f(x) < 0\}\right) = 0,$$

$L^p(\Omega, \mathcal{A}, \mu)$ becomes a Banach lattice. Sometimes, complexifications of real Banach lattices are necessary.

Definition B.2.2. Let F be a real Banach lattice, and define $F_{\mathbb{C}} := F \times F$, endowed with addition $(x, y)+(x', y') = (x+x', y+y')$, scalar multiplication $(a+ib)(x, y) = (ax-by, ay+bx)$ and norm

$$||(x, y)|| = \sup_{0 \leq \varphi \leq 2\pi} ||x \sin\varphi + y \cos\varphi||\,.$$

By identifying $(x, 0) \in F_{\mathbb{C}}$ with $x \in F$, we can write $F \subset F_{\mathbb{C}}$, and we will write $x + iy$ instead of (x, y). We will address $x \in F_{\mathbb{C}}$ as *positive* (writing $0 \leq x$) if and only if $x \in F_+$.

B.3 Positive operators

Using positivity on the Banach lattice, we can define positivity for operators on the Banach lattice.

Definition B.3.1. Let F, G be Banach lattices with positive cones F_+ and G_+ respectively. We refer to a linear operator $T : F \to G$ as *positive*, writing $T \geq 0$, if $Tx \in G_+$ for all $x \in F_+$. For two operators $S, T : F \to G$, we write $S \leq T$ in case $T - S \geq 0$.

Lemma B.3.1. *Let F, G be Banach lattices, and let S, T be positive operators $S, T : F \to G$. Then S, T are continuous (and thus, bounded), and for the operator norm, we have*

$$||T|| = \sup\{||Tx|| : \ x \in F_+, \ \ ||x|| = 1\},$$

and in case $0 \leq S \leq T$, we have $||S|| \leq ||T||$.

In the remainder of this section, we consider the operator algebra $\mathcal{R} = L(F)$, where F is a Banach lattice. By $\mathcal{R}_+$, we denote the set of all positive operators.

Example B.3.1. We give some examples.

- For $F = \mathbb{R}^d$ with some p-norm, we obtain $\mathcal{R} = \mathbb{R}^{d\times d}$ (with the corresponding operator norm) and
$$\mathcal{R}_+ = \left\{a = (a_{ij})_{i,j=1}^d : \ a_{ij} \geq 0\right\}.$$
- Let F be the space of all bounded real-valued sequences $(b_n)_{n\in\mathbb{N}_0}$, endowed with the supremum norm, and let positivity be defined by
$$b = (b_n)_{n\in\mathbb{N}_0} \geq 0 \ \Leftrightarrow \ \forall n \in \mathbb{N}_0 : \ b_n \geq 0.$$
Then,
$$\mathcal{R} = \left\{ a = (a_{ij})_{i,j\in\mathbb{N}_0} : \ a_{ij} \in \mathbb{R}, \ \ ||a|| = \sup_{i\in\mathbb{N}_0} \sum_{j\in\mathbb{N}_0} |a_{ij}| < \infty \right\}$$
and
$$\mathcal{R} = \{a = (a_{ij}) \in \mathcal{R} : \ a_{ij} \geq 0\}.$$
In particular, $a \in \mathcal{R}_+$ for any infinite substochastic matrix a.

We conclude with proving that positivity makes convergent series unconditionally convergent.

Theorem B.3.1. *Let F be a Banach lattice and let $\mathcal{R} = L(F)$, let $(a_n)_{n\in\mathbb{N}}$ be an F_+-valued or an $\mathcal{R}_+$-valued sequence, and let*

$$\sum_{n=1}^{\infty} a_n$$

converge to some $a \in F$ or $a \in \mathcal{R}$ respectively.

a) $\sum_{i\in\mathbb{N}} a_i$ *converges unconditionally.*

b) For $I_n \uparrow \mathbb{N}$, $\sum_{i\in I_n} a_i$ *converges unconditionally, and we have*

$$\lim_{n\to\infty} \sum_{i\in I_n} a_i = \sum_{i\in\mathbb{N}} a_i.$$

c) For a partition $\mathbb{N} = J_1 \dot\cup J_2 \dot\cup \ldots$, $\sum_{i\in J_n} a_i$ *converges unconditionally, and we have*

$$\sum_{i\in J_1} a_i + \sum_{i\in J_2} a_i + \ldots = \sum_{i\in\mathbb{N}} a_i.$$

Proof. Due to $a_i \geq 0$, $\min(J) > N$ implies $\sum\limits_{i\in J} a_i \leq \sum\limits_{i=N+1}^{\infty} a_i$, and hence, $\left\|\sum\limits_{i\in J} a_i\right\| \leq \left\|\sum\limits_{i=N+1}^{\infty} a_i\right\|$. Due to convergence, for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ with $\left\|\sum\limits_{i=N+1}^{\infty} a_i\right\| < \epsilon$, and for all $J \subset \mathbb{N}$ with $\min(J) > N$, we have $\left\|\sum\limits_{i\in J} a_i\right\| < \epsilon$. With these considerations, a) follows from Lemma A.1.1, and b) and c) follow as in the proof of Theorem A.1.1. □

Remark B.3.1. Note that by 'convergence', we always refer to norm-convergence. In function spaces, there are more ways of defining convergence. We will not discuss this topic extensively, but we remark that in Banach lattices, every weakly convergent increasing sequence is norm-convergent. In particular, this result applies to partial sums of series of positive summands.

Appendix C

Markov chains

In this appendix, we give an overview of some Markov processes, in particular, we consider Markov processes with discrete time or discrete state space.

C.1 Stochastic processes and the Markov property

We begin with summarizing some basic definitions for stochastic processes. For details, the reader is referred to [Kle06]. In our short discussion, we assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and that – for sake of simplicity – $I \subset \mathbb{R}$ is closed under addition and $\infty \in \overline{I}$.

- A *stochastic process* is a family $X = (X_t)_{t\in I}$ of random variables $X_t : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{E})$ where $(E, \mathcal{E})$ is a measurable space. $(E, \mathcal{E})$ is said to be the *state space*, and I is often interpreted as *time*.

- The *finite-dimensional distributions* of a stochastic process X are given by the distributions of vectors $(X_{t_1}, \ldots, X_{t_n})$, where $t_1, \ldots, t_n \in I$ with $t_1 < \ldots < t_n$. Since X itself can be interpreted as a random variable, it defines a probability distribution on the *sample space* $\left(E^I, E^{\otimes I}\right)$, and the images of X, that is the functions $t \mapsto X_t(\omega)$ for fixed $\omega \in \Omega$ are referred to as *samples* or *paths* of X.

- A family $\mathcal{F} = (\mathcal{F}_t)_{t\in I}$ of sub-σ-algebras for $(\Omega, \mathcal{A}, \mathbb{P})$ is said to be a *filtration* for $(\Omega, \mathcal{A}, \mathbb{P})$ if $s \leq t$ implies $\mathcal{F}_s \subset \mathcal{F}_t$.

- A stochastic process X on $(\Omega, \mathcal{A}, \mathbb{P})$ is said to be *adapted* to a filtration $\mathcal{F}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ if X_t is $\mathcal{F}_t$-measurable. The minimal filtration with this property is defined by $\mathcal{F}_t^X := \sigma\left((X_s)_{s\leq t}\right)$, and we refer to $\mathcal{F}^X$ as *canonical filtration*.

- Let $\mathcal{F}$ be a filtration and X be a stochastic process X. X is said to meet the *Markov property* with respect to $\mathcal{F}$ if

$$\forall s, t \in I \text{ with } s < t,\ \forall B \in \mathcal{E} :\ \mathbb{P}(X_t \in B|\mathcal{F}_s) = \mathbb{P}(X_t \in B|X_s).$$

  If we say that 'X meets the Markov property' without explicitly referring to some filtration, we assume $\mathcal{F} = \mathcal{F}^X$.

- X is said to be a *($\mathcal{F}$-)Markov process* if X meets the Markov property (with respect to $\mathcal{F}$), and if the transition probabilities are *time-homogeneous*, that is $\mathbb{P}(X_t \in B|X_s)$ is a function of $t-s$.
- For any stochastic process $X = (X_t)_{t\in I}$, by $\pi(t)$, we denote the distribution of t. If X_t converges in distribution (as $t \to \infty$) to some random variable X_∞, by $\pi(\infty)$, we denote its distribution. In this case, the probability measure $\pi(t)$ converges weakly to $\pi(\infty)$.

C.2 Discrete-time Markov processes

C.2.1 Basic dynamics

We continue with discrete-time Markov processes (we will use the abbreviation DTMP) $X = (X_n)_{n\in\mathbb{N}_0}$. For more details, the reader is referred to the book of Meyn and Tweedie [MT93]. Note that Meyn and Tweedie use the term 'Markov chain' for Markov processes with discrete time whereas our term DTMP emphasizes on the fact that the time is discrete whereas in principle, the state space is arbitrary. However, in what follows, we assume the state space to be a Polish space, and $\mathcal{E} = \mathcal{B}(E)$ to be the Borel-σ-algebra. This choice has the advantage that regular versions of conditional distribution exist, that is, conditional distributions can be represented by Markov kernels.

Following [MT93, section 3.4], let X be a DTMP with Polish state space $(E, \mathcal{E})$, and let $\mathbf{P} : E \times \mathcal{E}$ be characterized by $\mathbf{P}(x, B) = \mathbb{P}(X_1 \in B|X_0 = x)$, that is, $\mathbf{P}$ is a version of the distribution of $X_1|X_0$. Since E is Polish, $\mathbf{P}$ can be chosen as a *Markov kernel*, that is,

- $x \mapsto \mathbf{P}(x, B)$ is measurable for all $B \in \mathcal{E}$ and
- $B \mapsto \mathbf{P}(x, B)$ is a probability measure for all $x \in E$.

In a natural way, we can define the products

$$\begin{aligned} \mu\mathbf{P} \qquad &\text{by} \qquad \mu\mathbf{P}(B) = \int_E \mu(dx)P(x, B) \\ \mathbf{P}f \qquad &\text{by} \qquad \mathbf{P}f = \int_E P(x, dy)f(y) \end{aligned}$$

for σ-finite measures μ on $\mathcal{E}$ and measurable functions $f : E \to \mathbb{R}$ respectively. In particular, from one of these definitions, we obtain the product $\mathbf{P}\cdot\mathbf{P}$. Markov property and homogeneity yield that the Markov kernel $\mathbf{P}^n = \mathbf{P}\cdot\mathbf{P}^{n-1}$ is a (regular) version of the distribution of $X_n|X_0$. Simple implications are that

- $\pi(n) = \pi(0)\mathbf{P}^n$ is the distribution of X_n and
- any finite-dimensional distribution of X can be written in terms of $\pi(0)$ and $\mathbf{P}$.

On the other hand, by constructing the finite-dimensional distributions and applying Kolmogorov's Extension Theorem, for any Markov kernel $\mathbf{P}$ on $E \times \mathcal{E}$, and any probability measure $\pi(0)$, we are able to construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a stochastic process

X such that $\pi(0)$ is the initial distribution for X and $\mathbf{P}$ is its transition probability kernel. See [MT93, section 3.4] for more details.

Due to this construction, at least on Polish spaces, it makes sense that DTMPs X and a pair $(\pi(0), \mathbf{P})$ are often used synonymously. Due to $\pi(n) = \pi(0)\mathbf{P}^n$, the theory of DTMPs and the theory of semigroups $(\mathbf{P}^n)_{n\in\mathbb{N}_0}$ are quite strongly related. $\mathbf{P}$ is said to be the *transition probability kernel* for the DTMP, and $\pi(0)$ is referred to as *initial distribution*.

For a DTMP $X = (X_n)_{n\in\mathbb{N}_0}$, we may 'change the initial distribution' without losing the Markov property or the homogeneity. This change only affects the probability measure $\mathbb{P}$ of the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$. In particular, we may choose the Dirac–measure δ_x as the new initial distribution. The corresponding probability measure on $(\Omega, \mathcal{A})$ will be denoted by $\mathbb{P}_x$. In case of $\mathbb{P}(X_0 = x) > 0$, we have

$$\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | X_0 = x).$$

Even in case that $\pi(0)$ is substochastic or $\mathbf{P}$ is sub-Markovian, we may identify the pair $(\pi(0), \mathbf{P})$ with some 'probability-losing' DTMP:

- Define $\overline{E} = E \cup \{\infty\}$ with some state $\infty \notin E$, and extend $\mathcal{B}(E)$ appropriately to $\mathcal{B}\left(\overline{E}\right)$.
- Define $\overline{\pi}(0)$ by $\overline{\pi}(0)(A) = \pi(0)(A)$ for $A \in \mathcal{B}(E)$ and $\overline{\pi}(0)(\{\infty\}) = 1 - \pi(0)(E)$.
- Define $\overline{\mathbf{P}}(x, A) = \mathbf{P}(x, A)$ for $x \in E$, $A \in \mathcal{B}(E)$, and $\overline{\mathbf{P}}(x, \{\infty\}) = 1 - \mathbf{P}(x, E)$ for $x \in E$.
- Define $\overline{\mathbf{P}}(\infty, A) = \mathbf{1}_A(\infty)$ for all $A \in \mathcal{B}\left(\overline{E}\right)$.
- Then there is some DTMP X with initial distribution $\overline{\pi}(0)$ and transition kernel $\overline{\mathbf{P}}$.
- In particular, $\mathbb{P}(X_n \in A) = \left(\overline{\pi}(0)\overline{\mathbf{P}}^n\right)(A)$ for $A \in \mathcal{B}\left(\overline{E}\right)$. Due to construction of $\overline{\pi}(0)$ and $\overline{\mathbf{P}}$, we have $\mathbb{P}(X_n \in A) = \left(\pi(0)\mathbf{P}^n\right)(A)$ for $A \in \mathcal{B}(E)$.
- Hence, we interprete X as DTMP with initial distribution $\pi(0)$ and transition kernel $\mathbf{P}$, and allow $\mathbb{P}(X_n \notin E) = 1 - \left(\pi(0)\mathbf{P}^n\right)(E) > 0$.

C.2.2 Hitting probabilities and first hitting times

Let X be a DTMP with values in $(E, \mathcal{B}(E))$ and let $C \in \mathcal{B}(E)$. Then the

- *first return time* to C is given by $\tau_C := \inf\{n \in \mathbb{N} : \ X_n \in C\}$,
- *first hitting time* on C is given by $\sigma_C := \inf\{n \in \mathbb{N}_0 : \ X_n \in C\}$.

$L(x, C) = \mathbb{P}_x(\tau_C < \infty)$ is the probability of ever returning to set C when starting in x. This probability will be essential for defining recurrence. For discussing hitting probabilities, we focus on the probability of ever being in set C, that is $\mathbb{P}_x(\sigma_C < \infty)$ which is trivially 1 for $x \in C$.

Theorem C.2.1. *Let X be a DTMP with values in $(E, \mathcal{E})$ and transition probability kernel $\mathbf{P}$, let $C \in \mathcal{E}$, and let $g : E \to \mathbb{R}$ be defined by $g(x) = \mathbb{P}_x(\sigma_C < \infty)$. Then $g(x) = 1$ for $x \in C$ and*

$$\int_E \mathbf{P}(x, dy)g(y) = g(x), \qquad x \in E \setminus C.$$

Furthermore, for any bounded measurable solution $h : E \to \mathbb{R}_{\geq 0}$ of

$$\begin{aligned} h(x) &\geq 1, \qquad x \in C, \\ h(x) &\geq \int_E \mathbf{P}(x, dy)h(y), \qquad x \in E \setminus C, \end{aligned}$$

we have $h(x) \geq g(x)$ for all $x \in E$.

Proof. See [MT93, Proposition 8.4.1] for details, here we give the sketch: $g(x) = 1 \leq h(x)$ for $x \in C$ is obvious, and $\int \mathbf{P}(x, dy)g(y) = g(x)$ for $x \in E \setminus C$ is a direct consequence of the Markov property. A simple induction yields

$$h(x) \geq \sum_{n=1}^{N} \int_C \mathbb{P}_x(X_n \in dy, \sigma_C = n)h(y) + \int_{E \setminus C} \mathbb{P}_x(X_N \in dy, \sigma_C > N)h(y),$$

and $N \to \infty$ yields $h(x) \geq g(x)$. □

We generalize this concept, and prove

Theorem C.2.2. *Let X be a DTMP with values in $(E, \mathcal{E})$ and transition probability kernel $\mathbf{P}$, and let $C \in \mathcal{E}$ and let $\mathcal{E}_C = \{B \cap C : B \in \mathcal{E}\}$. Then a kernel (measurable in the first variable, measure in the second) $G : E \times \mathcal{E}_C \to \mathbb{R}$ is defined by $G(x, B) = \mathbb{P}_x(\sigma_C < \infty, X_{\sigma_C} \in B)$. For $x \in C$, we have $G(x, B) = \mathbf{1}_B(x)$, and for $x \notin C$, we have*

$$G(x, B) = \int_E \mathbf{P}(x, dy)G(y, B).$$

For any other function $H : E \times \mathcal{E}_C \to \mathbb{R}_{\geq 0}$ which is measurable in the first variable for fixed B, and which meets

$$\begin{aligned} H(x, B) &\geq \mathbf{1}_B(x), \qquad x \in C, \\ H(x, B) &\geq \int_E \mathbf{P}(x, dy)H(y, B), \qquad x \in E \setminus C, \end{aligned}$$

we have $H(x, B) \geq G(x, B)$ for all $x \in E$, $B \in \mathcal{E}_C$.

Proof. For $x \in C$, $G(x, B) = \mathbf{1}_B(x) \leq H(x, B)$ is a direct consequence of the definition of σ_C. For $x \notin C$, we write

$$G(x, B) = \mathbf{P}(x, B \cap C) + \int_{E \setminus C} \mathbf{P}(x, dy)\mathbb{P}_y(\sigma_C < \infty, X_{\sigma_C}) = \int_E \mathbf{P}(x, dy)G(y, B).$$

Furthermore, for $x \notin C$, we obtain inductively

$$\begin{aligned} H(x,B) &\geq \int_C \mathbf{P}(x,dy)H(y,B) + \int_{E\setminus C} \mathbf{P}(x,dy)H(y,B) \\ &\geq \int_C \mathbf{P}(x,dy)H(y,B) + \int_{E\setminus C} \mathbf{P}(x,dy)\left(\int_C \mathbf{P}(y,dz)H(z,B) + \int_{E\setminus C} \mathbf{P}(y,dz)H(z,B)\right) \\ &= \int_C \sum_{n=1}^{2} \mathbb{P}_x(\sigma_C = n, X_n \in dy)H(y,B) + \int_{E\setminus C} \mathbb{P}_x(\sigma_C > 2, X_2 \in dy)H(y,B) \\ &\geq \dots \\ &\geq \int_C \sum_{n=1}^{N} \mathbb{P}_x(X_n \in dy, \sigma_C = n)H(y,B) + \int_{E\setminus C} \mathbb{P}_x(X_N \in dy, \sigma_C > N)H(y,B). \end{aligned}$$

Since $H(y,B) \geq \mathbf{1}_B(y)$ for $y \in C$, and due to $H(y,B) \geq 0$, we obtain

$$H(x,B) \geq \sum_{n=1}^{N} \mathbb{P}_x(X_n \in B, \sigma_C = n) \overset{N\to\infty}{\to} G(x,B).$$

□

Next, we give a result on $\mathbb{E}_x\left[\sum_{k=0}^{\sigma_C-1} f(X_k)\right]$. For $f = \mathbf{1}$, we obtain mean first hitting times.

Theorem C.2.3. *Let X be a DTMP with values in $(E,\mathcal{E})$ and transition probability kernel $\mathbf{P}$, and let $C \in \mathcal{E}$, $f : E \to [0,\infty)$ measurable. For any measurable function $h : E\setminus C \to [0,\infty]$ with*

$$h(x) \geq \int_{E\setminus C} \mathbf{P}(x,dy)h(y) + f(x), \qquad x \in E \setminus C,$$

we have $g(x) := \mathbb{E}_x\left[\sum_{k=0}^{\sigma_C-1} f(X_k)\right] \leq h(x)$ and

$$g(x) = \int_{E\setminus C} \mathbf{P}(x,dy)g(y) + f(x)$$

for all $x \in E \setminus C$. In particular, if $h(x) < \infty$ for some $x \in E \setminus C$, we have $g(x) < \infty$.

Proof. For $x \notin C$, the relationship

$$g(x) = f(x) + \int_{E\setminus C} \mathbf{P}(x,dy)\mathbb{E}_y\left[\sum_{k=0}^{\sigma_C-1} f(X_k)\right] = f(x) + \int_{E\setminus C} \mathbf{P}(x,dy)g(y)$$

is a direct consequence of Markov property and homogeneity. The condition on h implies $h(x) \geq f(x)$ for all $x \in E \setminus C$, and an easy induction yields

$$g(x) \geq \mathbb{E}_x\left[\sum_{k=0}^{\min\{\sigma_C-1,N\}} f(X_k)\right]$$

for all $x \in E \setminus C$ and for all $N \in \mathbb{N}$. Taking $N \to \infty$, by monotone convergence, we obtain $h(x) \geq g(x)$ for $x \in E\setminus C$ (see [MT93, Theorem 14.2.2] for a similar statement and proof). □

In the theory of DTMPs, these results are quite helpful when deriving drift criteria for recurrence, positive recurrence or f-positivity. Direct practical applications concern the computation of absorption probabilities and mean absorption times.

C.2.3 φ-irreducibility, small sets and petite sets

Before we classify DTMPs (recurrence, transience, ...), we introduce some helpful terms and concepts. Again, let X be a DTMP with values in a Polish state space $(E, \mathcal{E})$ and transition probability kernel $\mathbf{P}$. For the concept of irreducibility, we refer to [MT93, section 4.2], for small and petite sets, we refer [MT93, section 5.2] and [MT93, section 5.5] respectively.

- A non-trivial σ-finite measure φ on $\mathcal{E}$ is said to be an *irreducibility measure* for X if $\varphi(B) > 0$ implies $L(x, B) := \mathbb{P}_x(\tau_B < \infty) > 0$ for all $x \in E$. If X admits an irreducibility measure φ, X is referred to as φ-irreducible.

- φ is said to be a *maximal irreducibility measure*, if for any other irreducibility measure φ', we have $\varphi'(B) > 0 \Rightarrow \varphi(B) > 0$ (briefly: $\varphi' \prec \varphi$). If X admits an irreducibility measure, it admits a maximal irreducibility measure, and maximal irreducibility measures are unique up to equivalence, that is, if φ and φ' are maximal irreducibility measures, we have $\varphi \prec \varphi'$ and $\varphi' \prec \varphi$. *Remark:* Note that Meyn and Tweedie implicitely distinguish between arbitrary irreducibility measures and maximal irreducibility measures by denoting maximal irreducibility measures by ψ. We do not follow this approach. If we want φ to be a maximal irreducibility measure, we will explicitly write this.

- If X is φ-irreducible with some maximal irreducibility measure φ, we define $\mathcal{E}^+ := \{B \in \mathcal{E} : \ \varphi(B) > 0\}$. Since maximal irreducibility measures are unique up to equivalence, $\mathcal{E}^+$ is well-defined. $\mathcal{E}^+$ tells us where the DTMP actually 'lives'.

- A set $C \in \mathcal{E}$ is said to be *small* if there is some $m \in \mathbb{N}$, some $\epsilon > 0$ and some probability measure ν on $\mathcal{E}$ such that

$$\forall x \in C, \ B \in \mathcal{E} : \ \mathbf{P}^m(x, B) \geq \epsilon \nu(B).$$

- If X is φ-irreducible, every $D \in \mathcal{E}^+$ contains a small set $C \subset D$ with $C \in \mathcal{E}^+$. In particular, there is some small set $C \in \mathcal{E}^+$. Furthermore, $E = \bigcup_{i=1}^{\infty} C_i$ with small sets C_i.

- A set $C \in \mathcal{E}$ is said to be *petite* if there is some probability vector $a = (a_n)_{n \in \mathbb{N}_0}$ on $\mathbb{N}_0$, some $\epsilon > 0$ and some probability measure ν on $\mathcal{E}$ such that

$$\forall x \in C, \ B \in \mathcal{E} : \ K_a(x, B) := \sum_{n=0}^{\infty} a_n \mathbf{P}^n(x, B) \geq \epsilon \nu(B).$$

- Obviously, small sets are petite. Hence, in case of φ-irreducibility, there is always a petite set $C \in \mathcal{E}^+$. Furthermore, for φ-irreducible DTMPs, the union of finitely many petite sets is petite again, and hence, we can write $E = \bigcup_{i=1}^{\infty} C_i$ with petite sets $C_1 \subset C_2 \subset C_3 \subset \ldots$.

C.2.4 Classifying DTMPs

Next, we turn to defining (Harris) recurrence, (uniform) transience and (a)periodicity.

- A set $D \in \mathcal{E}$ is said to be *absorbing* if $\mathbf{P}(x, D) = 1$ for all $x \in D$, see [MT93, section 4.2].
- Let C be small, that is $\mathbf{P}^m(x, B) \geq \epsilon\nu(B)$ for all $x \in C$, $B \in \mathcal{E}$. Define

  $$M_C = \{n \in \mathbb{N} : \ \exists \epsilon_n > 0 \ \forall x \in C, B \in \mathcal{E} \ \ \mathbf{P}^n(x, B) \geq \epsilon_n \nu(B)\} \quad \text{and} \quad d_C = \gcd M_C.$$

  The set M_C is closed under addition, and d_C is said to be the *period* of C. In case of $d_C = 1$, C is said to be *aperiodic*.
- Let X be φ-irreducible. A *d-cycle* is a family $\{E_0, \ldots, E_{d-1}\}$ of pairwise disjoints sets from $\mathcal{E}$ such that
  - $\mathbf{P}(x, E_{i+1}) = 1$ for $x \in E_i$ $(i \mod d)$ and
  - $E \setminus \bigcup_{i=0}^{d-1} E_i \notin \mathcal{E}^+$.

  The largest d for which a d-cycle occurs is said to be the *period* of X. In case $d = 1$, X is referred to as *aperiodic*.
- Let X be φ-irreducible and $C \in \mathcal{E}^+$ small. Then $d = d_C$, that is, the periodic behaviour of small sets is 'representative' for the periodic behaviour of the complete DTMP. In particular, $d < \infty$. See [MT93, section 5.4] for more details on the cyclic behaviour.
- Let η_B be the *occupation time* for set $B \in \mathcal{E}$, that is, $\eta_B = \sum_{n=1}^{\infty} \mathbf{1}_B(X_n)$ and

  $$U(x, B) = \mathbb{E}_x [\eta_B] = \sum_{n=1}^{\infty} \mathbf{P}^n(x, B).$$

  If $U(x, B) \leq M$ for all $x \in B$, B is said to be *uniformly transient*. In this case, $U(x, B) \leq M + 1$ for all $x \in E$.
- If B can be covered by a countable number of uniformly transient sets, that is $B = \bigcup_{i=1}^{\infty} T_i$ with uniformly transient T_i, B is referred to as *transient*. In particular, if E is transient, we refer to X as *transient*, see [MT93, section 8.2].
- If X is φ-irreducible and transient, every petite set is uniformly transient [MT93, Theorem 8.3.5].
- If D is absorbing and $L(x, D) = 1$ for all $x \in E \setminus D$, $E \setminus D$ is transient [MT93, Theorem 8.3.6].
- If $U(x, B) = \infty$ for all $x \in B$, B is said to be *recurrent*. Note that there may be sets B which are neither recurrent nor transient, and that there may be sets B which are both. Recurrent sets cannot be uniformly transient, see [MT93, section 8.2].

- If X is φ-irreducible and every $B \in \mathcal{E}^+$ is recurrent, X is said to be *recurrent*. Every φ-irreducible DTMP X is either recurrent or transient [MT93, section 8.2].
- If $L(x, B) = 1$ for all $x \in B$, B is said to be *Harris recurrent*. If X is φ-irreducible and every $B \in \mathcal{E}^+$ is Harris recurrent, X is referred to as Harris recurrent. In both situations, Harris recurrence implies recurrence, whereas the converse is not true in general, see [MT93, section 9.1].
- Let X be φ-irreducible and recurrent. Then we can decompose $E = H \cup N$ where H is absorbing, the restriction of X to H is $\varphi|_H$-irreducible and Harris recurrent, and $N \notin \mathcal{E}^+$ [MT93, Theorem 9.1.5]. In this situation $E = H \cup N$ is referred to as *Harris decomposition*.
- Easy criteria for (Harris) recurrence involve petite sets: Let X be φ-irreducible, let C be petite, and let $L(x, C) = 1$ for all $x \in C$ ($x \in E$). Then X is (Harris) recurrent [MT93, Theorem 8.3.6 and Proposition 9.1.7].

We add another criterion for uniform transience here:

Lemma C.2.1. *Let X be φ-irreducible, let $D \in \mathcal{E}$ be absorbing, and let $C \in E \setminus D$ be petite. Then C is uniformly transient.*

Proof. Due to $L(x, E \setminus D) = 0$ for $x \in D$, we have $\varphi(E \setminus D) = 0$ for any irreducibility measure φ. The dynamics on D has no impact on the classification of $C \in E \setminus D$, and hence, we may redefine $D = \{x_0, x_1, \ldots\}$, $\mathbf{P}(x, \{x_0\}) = \mathbf{P}(x, D)$ for $x \in E \setminus D$ and $\mathbf{P}(x_n, x_{n+1}) = \frac{3}{4}$ and $\mathbf{P}(x_{n+1}, \{x_n\}) = \frac{1}{4}$ for $n \in \mathbb{N}_0$. The new DTMP is still φ'-irreducible, where we may choose $\varphi'(B) = |\{n \in \mathbb{N} : \ x_n \in B\}|$, and it is quite easy to see that it is transient. As a petite set, C is uniformly transient. □

C.2.5 Invariant and subinvariant measures

Invariant measures are quite important for the asymptotic analysis of DTMPs (see below). For some considerations, it is helpful to introduce the more general concept of subinvariant measures. Again, X is assumed to be a DTMP with values in the Polish state space $(E, \mathcal{E})$ and transition probability kernel $\mathbf{P}$. For details, we refer to [MT93, chapter 10].

- A *subinvariant measure* is a non-trivial, σ-finite measure ψ on $\mathcal{E}$ with $\psi\mathbf{P} \leq \psi$. In case of equality, ψ is an *invariant measure*.
- For subinvariant measures ψ, we have $\psi(C) < \infty$ for all petite sets $C \in \mathcal{E}$.
- Finite subinvariant measures are invariant. An invariant probability measure π is referred to as *invariant distribution*.
- If X is φ-irreducible and admits an invariant distribution π, X is recurrent. Hence, it makes sense to define X as *positive recurrent* if there is an invariant distribution. If X is Harris recurrent and positive recurrent, X is said to be *positive Harris*. In this situation, π is a maximal irreducibility measure.

- If X is φ-irreducible and recurrent, all subinvariant measures are invariant and only differ by constant multiples. In case of positive recurrence, the invariant distribution π is unique and it is a maximal irreducibility measure.
- Similar as recurrence, positive recurrence can be characterized in terms of petite sets: Let X be φ-irreducible, let $C \in \mathcal{E}$ be petite, and let $\sup_{y \in C} \mathbb{E}_y [\tau_C] < \infty$. Then X is positive recurrent [MT93, Theorem 11.3.15].

For our purposes, we need a kernel for which all components are subinvariant up to modifications on some set C.

Theorem C.2.4. *Let X be a DTMP, let $C \in \mathcal{E}$ and let*

$$F(x, B) := \mathbb{E}_x \left[\sum_{n=0}^{\tau_C - 1} \mathbf{1}_B(X_n) \right] = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n \in B, \tau_C > n), \qquad x \in C,\ B \in \mathcal{E}$$

and

$$T(x, B) = \int_E F(x, dy) \mathbf{P}(y, B), \qquad x \in C,\ B \in \mathcal{E}_C.$$

a) $F : C \times \mathcal{E} \to [0, \infty]$ is a kernel with

$$\begin{aligned} F(x, B) &= \mathbf{1}_B(x), \qquad B \subset C, \\ F(x, B) &= \int_E F(x, dy) \mathbf{P}(y, B), \qquad B \subset E \setminus C. \end{aligned}$$

b) If $H : C \times \mathcal{E} \to [0, \infty]$ is a function for which $H(x, \cdot)$ defines a measure on $\mathcal{E}$ for all $x \in C$, and for which

$$\begin{aligned} H(x, B) &\geq \mathbf{1}_B(x), \qquad B \subset C, \\ H(x, B) &\geq \int_E H(x, dy) \mathbf{P}(y, B), \qquad B \subset E \setminus C \end{aligned}$$

holds, then we have $H \geq F$ componentwise, that is, $H(x, B) \geq F(x, B)$ for all $x \in C$, $B \in \mathcal{E}$. In particular, if $H(x, \cdot)$ is σ-finite for all $x \in C$, F is a σ-finite kernel.

c) If $L(x, C) > 0$ for all $x \in E \setminus C$ (for φ-irreducible X, this corresponds to $C \in \mathcal{E}^+$), F is σ-finite.

d) $T : C \times \mathcal{E}_C$ is a sub-Markovian kernel with

$$T(x, B) := \int_E F(x, dy) \mathbf{P}(y, B) = \int_E \mathbf{P}(x, dy) G(y, B) = \mathbb{P}_x(\tau_C < \infty, X_{\tau_C} \in B).$$

It is a Markov kernel if and only if C is Harris recurrent. In this case, T is the transition probability kernel for the C-censored Markov process, that is, X is sampled whenever $X_n \in C$.

e) Let ψ_C be a (sub)invariant measure for T, and let F be σ-finite. Then $\psi := \psi_C F$ is a (sub)invariant measure for $\mathbf{P}$.

f) If X is φ-irreducible, T is $\varphi|_C$-irreducible.

Proof. a) $F(x,B) = \mathbf{1}_B(x)$ for $B \subset C$ is obvious. For $B \subset E \setminus C$, we have

$$\begin{aligned}\int_E F(x,dy)\mathbf{P}(y,B) &= \mathbf{P}(x,B) + \int_{E\setminus C} \sum_{n=1}^{\infty} \mathbb{P}_x(X_n \in dy, \tau_C > n)\mathbf{P}(y,B) \\ &= \mathbf{P}(x,B) + \sum_{n=2}^{\infty} \mathbb{P}_x(X_n \in B, \tau_C > n) = F(x,B).\end{aligned}$$

b) $F(x,B) = \mathbf{1}_B(x) \le H(x,B)$ for $B \subset C$ is clear, and for $B \subset E \setminus C$, we prove

$$H(x,B) \ge \sum_{n=1}^{N} \mathbb{P}_x(X_n \in B, \tau_C > n)$$

by induction: For $N = 1$, we have

$$H(x,B) \ge \int_C H(x,dy)\mathbf{P}(y,B) = \mathbf{P}(x,B),$$

and for $N \ge 2$, we use

$$\begin{aligned}H(x,B) &\ge \mathbf{P}(x,B) + \int_{E\setminus C} H(x,dy)\mathbf{P}(y,B) \\ &\ge \mathbf{P}(x,B) + \int_{E\setminus C} \sum_{n=1}^{N-1} \mathbb{P}_x(X_n \in dy, \tau_C > n)\mathbf{P}(y,B) \\ &= \mathbf{P}(x,B) + \sum_{n=2}^{N} \mathbb{P}_x(X_n \in B, \tau_C > n) \\ &= \sum_{n=1}^{N} \mathbb{P}_x(X_n \in B, \tau_C > n) \overset{N\to\infty}{\to} F(x,B).\end{aligned}$$

c) For $B \subset E \setminus C$, we write

$$\begin{aligned}F(x,B) &= \sum_{n=1}^{\infty} \mathbb{P}_x(X_n \in B, \tau_C > n) \\ &= \int_{E\setminus C} \mathbf{P}(x,dy) \sum_{n=0}^{\infty} \mathbb{P}_y(X_n \in B, \tau_C > n) \\ &= \int_{E\setminus C} \mathbf{P}(x,dy) \sum_{n=0}^{\infty} \overline{\mathbf{P}}^n(y,B),\end{aligned}$$

where we put $\overline{\mathbf{P}}(y,B) = \mathbf{P}(y,B)$ for $y \in E \setminus B$ and $\overline{\mathbf{P}}(y,B) = \mathbf{1}_B(y)$ for $y \in C$. With respect to $\overline{\mathbf{P}}$, C is absorbing, and due to $L(x,C) > 0$ for all $x \in E \setminus C$, $E \setminus C$ is transient

with respect to $\overline{\mathbf{P}}$. By definition, we can write $E \setminus C = \bigcup_{i=1}^{\infty} B_i$ with uniformly transient sets B_i (with respect to $\overline{\mathbf{P}}$). Then $\sum \overline{\mathbf{P}}^n(y, B_i) \leq M_i$ for all $y \in E$, and hence,

$$F(x, B_i) \leq \int_{E \setminus C} \mathbf{P}(x, dy) M_i \leq M_i < \infty.$$

Obviously, $F(x, C) = 1 < \infty$, and due to $E = C \cup \bigcup_{i=1}^{\infty} B_i$, F is σ-finite.

d) The representations of T are easy to see. The last one yields $T(x, C) = \mathbb{P}_x(\tau_C < \infty) \leq 1$ with $= 1$ for all $x \in C$ if and only if C is Harris recurrent.

e) If ψ_C and F are σ-finite, the product $\psi_C F$ is σ-finite again. For $B \subset C$, we have $FP(x, B) = T(x, B)$, and for $B \subset E \setminus C$, we have $FP(x, B) = F(x, B)$. In total, these considerations yield $FP(x, B) = T(x, B \cap C) + F(x, B \cap (E \setminus C))$. Thus, $\psi_C T \overset{=}{\leq} \psi_C$ implies

$$\psi \mathbf{P} = \psi_C F \mathbf{P} = \psi_C T(\cdot \cap C) + \psi_C F(\cdot \cap (E \setminus C)) \overset{=}{\leq} \psi_C(\cdot \cap C) + \psi_C F(\cdot \cap (E \setminus C)) = \psi_C F = \psi.$$

f) The statement follows immediately from the interpretation of K as transition probability kernel for the censored DTMP: If there is a positive probability for reaching B from x, this is also true for the censored DTMP.

□

C.2.6 Limiting behaviour

Invariant measures are quite important for the limit behaviour of DTMPs, due to the following famous asymptotic results:

Theorem C.2.5. *Let X be φ-irreducible.*

- *Let X be positive Harris and aperiodic, let π be the invariant distribution, and let $||\cdot||_{TV}$ denote the total variation norm for signed measures. Then $||\pi(n) - \pi||_{TV} \overset{n \to \infty}{\to} 0$, that is, for any bounded and measurable function $f : E \to \mathbb{R}$, we have*

$$\lim_{n \to \infty} ||\pi(n) f - \pi f|| = 0.$$

  In particular, $\pi(n)$ converges weakly to π, or equivalently, X_n converges in distribution to some random variable X_∞ with distribution π [MT93, Theorem 13.3.3].

- *Let X be positive Harris and d-periodic, and let π be the invariant distribution. Then*

$$\lim_{n \to \infty} \left|\left| \frac{1}{d} \sum_{c=0}^{d-1} \pi(nd + c) - \pi \right|\right| = 0,$$

  see [MT93, Theorem 13.3.4].

- *Let X be Harris recurrent, and let ψ denote an invariant distribution, let $f, g : E \to \mathbb{R}$ be measurable and ψ-integrable, that is $\psi|f| < \infty$ and $\psi|g| < \infty$, and let $\psi g \neq 0$. Then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(X_k)}{\sum_{k=0}^{n-1} g(X_k)} = \frac{\psi f}{\psi g} \qquad \text{almost surely,}$$

  see [MT93, Theorem 17.3.2].

- *For $g = 1$, we obtain: Let X be positive Harris, let π denote the invariant distribution, and let f be π-integrable. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \pi f \qquad \text{almost surely.}$$

- *Let X be (positive) recurrent, but not necessarily Harris. Then there is some set $N \notin \mathcal{E}^+$ such that whenever $\pi(0)(N) = 0$, the above results hold (follows from the Harris decomposition, see [MT93, Theorem 13.3.4(ii)]).*

C.2.7 Drift criteria

For practical applications, drift criteria are useful for deciding whether or not a DTMP is (Harris) recurrent or even positive recurrent.

Theorem C.2.6. *Let X be a φ-irreducible DTMP with values in $(E, \mathcal{E})$ and transition probability kernel $\mathbf{P}$, let $g : E \to [0, \infty)$ be measurable, define the corresponding* drift *$d_g : E \to \mathbb{R} \cup \{\infty\}$ by*

$$d_g(x) = (\mathbf{P}g - g)(x) = \int_E \mathbf{P}(x, dy)g(y) - g(x) = \mathbb{E}_x\left[g(X_1) - g(X_0)\right],$$

let $C \in \mathcal{E}$ be petite, let $\epsilon, b > 0$, and let $f : E \to [0, \infty)$ be measurable.

- *Let g be unbounded off petite sets, that is $\{x \in E : \ g(x) \leq r\}$ is petite for all $r > 0$, and let $d_g(x) \leq 0$ for $x \in E \setminus C$. Then X is Harris recurrent [MT93, Theorem 9.1.8].*

- *Let g be bounded on C, and let $d_g(x) \leq -\epsilon + b\mathbf{1}_C(x)$ for all $x \in E$. Then X is positive Harris [MT93, Theorem 11.3.15 or Theorem 11.0.1].*

C.2.8 Special case: DTMCs

An important special case of DTMPs is obtained from assuming $(E, \mathcal{E})$ to be discrete, that is, E is finite or countably infinite, and $\mathcal{E}$ is the power set. In this situation, we refer to X as *discrete-time Markov chain*, abbreviated by DTMC.

- The distributions $\pi(n)$ of X_n are completely characterized by the counting densities $\pi_i(n) = \pi(n)(\{i\})$, similarly, we can argue for any measure on $\mathcal{E}$, e.g. invariant measures, irreducibiliity measures, ...
- The transition probability kernel $\mathbf{P}$ may be identified with the (stochastic) *transition probability matrix* $\mathbf{P} = (p_{ij})$, where $p_{ij} = \mathbf{P}(i, \{j\})$. The kernel product corresponds to the matrix product. For the n-step transition probabilities, we write $\mathbf{P}^n = \left(p_{ij}^{(n)}\right)_{i,j\in E}$.
- G, F, K may be represented by matrices, too.
- In case of $L(i, \{j\}) > 0$, we write $i \to j$. If $i \to j$ and $j \to i$, we write $i \leftrightarrow j$ and say that i and j *communicate*.
- If we refer to a DTMC X as *irreducible* without specifying the measure, we assume X to be ζ-irreducible, where ζ is the counting measure on $\mathcal{E}$. Hence, X is irreducible if and only if all states communicate. In this situation, any non-empty set is in $\mathcal{E}^+$.
- Every finite set is petite.
- The period d_i of some state $i \in E$ is $d_i = \gcd\left\{n \in \mathbb{N} : \ p_{ii}^{(n)} > 0\right\}$. If i and j communicate, $d_i = d_j$. In particular, if X is irreducible, all states have the same period d, and we finde a d-cycle $\{E_0, \dots, E_{d-1}\}$ with $E = \bigcup_{i=0}^{d-1} E_i$.
- For single states, recurrence and Harris recurrence are equivalent, that is, i is recurrent $\Leftrightarrow \ U(i, \{i\}) = \infty \ \Leftrightarrow \ L(i, \{i\}) = 1$. Otherwise, i is transient (and $\{i\}$ is uniformly transient). Finite sets of transient states are still uniformly transient.
- If i and j communicate, either both are recurrent or both are transient. In particular, in case of irreducibility, either all states (and X itself) are recurrent (and then Harris recurrent due to $N = \emptyset$ in the Harris decomposition $E = H \cup N$) or transient.
- For single states, positive recurrence may be defined by $\mathbb{E}_i[\tau_i < \infty] < \infty$. Again, if i and j communicate, either both are positive recurrent or both are not positive recurrent. In case of irreducibility, positive recurrence of one state implies positive recurrence of all states and the existence of an invariant distribution.
- In the drift criteria, we may replace the petite set C by some finite set C.

In total, for DTMCs, we could have made things much easier (irreducibility instead of φ-irreducibility, no distinguishing between recurrence and Harris recurrence, no distinguishing between uniform transience and transience, no need for petite sets, ...). Nevertheless, the general results for φ-irreducible DTMPs extend some of the very popular results for irreducible DTMCs. As an easy example (similar to that in [MT93, section 9.1.2]), we consider the DTMC with state space $E = \mathbb{N}_0$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & & & & \\ 1-\beta_1 & 0 & \beta_1 & & \\ 1-\beta_2 & & & \beta_2 & \\ \vdots & & & & \ddots \end{pmatrix},$$

where $0 < \prod_{i=1}^{\infty} \beta_i < 1$. State 0 is absorbing and recurrent, all other states are transient. Whereas the set $\{0,1\}$ is recurrent, it is not Harris recurrent, since $L(1,\{0,1\}) = 1 - \prod_{i=1}^{\infty} \beta_i < 1$. Furthermore, we have no ζ-irreducibiility, but we have δ_0-irreducibility where δ_0 is the Dirac-measure in 0, and the restriction of X to $\{0\}$ is obviously Harris recurrent and $(1,0,0,\ldots)$ is an invariant distribution. If starting in 0, the limit theorems still hold. This example becomes more interesting when replacing the single state 0 by some set E_0 with $E_0 \cap \mathbb{N} = \emptyset$. If E_0 is φ-irreducible and Harris recurrent, the above considerations are similar, and the limit theorems still hold whenever $X_0 \in E_0$.

C.3 Continuous-time Markov chains

A continuous-time Markov chain (CTMC) is a Markov process (Markov property, homogeneity) $Y = (Y_t)_{t\geq 0}$ with discrete state space $(E, \mathcal{E})$, that is, E is countable and $\mathcal{E}$ is the power set. As for DTMCs, the transient distributions are uniquely characterized by their counting densities, and hence, we identify $\pi(t) = (\pi_i(t))_{i\in E}$, where $\pi_i(t) = \mathbb{P}(X_t = i)$. With some effort, it can be shown that any CTMC has a version with càdlàg paths (continuous from the right, limits from the left exist), see [And91, section 1.1] and references therein. Hence, in what follows, we assume the paths of X to be càdlàg. Up to notation, we follow the book of Anderson [And91].

C.3.1 Basic dynamics

We interprete $\pi(t)$ as a row vector, and introduce the *transition probability function* $\mathbf{P}(t) = (p_{ij}(t))_{i,j\in E}$ where $p_{ij}(t) = \mathbb{P}_i(Y_t = j) = \mathbb{P}(Y_t = j | Y_0 = i)$. Total probability directly yields

$$\begin{aligned} \pi(t) &= \pi(0)\mathbf{P}(t), \qquad t \geq 0, \\ \mathbf{P}(s+t) &= \mathbf{P}(s)\mathbf{P}(t), \qquad s,t \geq 0 \qquad \text{(Chapman-Kolmogorov)}. \end{aligned}$$

The initial distribution $\pi(0)$ and the transition probability function $t \mapsto \mathbf{P}(t)$ characterize the complete dynamics of the CTMC. Any function $t \mapsto \mathbf{P}(t)$ with

- $\mathbf{P}(t)$ is a substochastic matrix for all $t \geq 0$,
- $\mathbf{P}(0)$ is the identity, $\mathbf{P}(0) = I$,
- $\mathbf{P}(s+t) = \mathbf{P}(s)\mathbf{P}(t)$ for all $s,t \geq 0$

is said to be a *transition probability function*, and in this situation, the family $\{\mathbf{P}(t) : t \geq 0\}$ is said to be the *transition semigroup*. $\mathbf{P}$ is said to be a *standard transition probability function* if $\mathbf{P}(0+) = \mathbf{P}(0) = I$. For any stochastic transition probability function $t \mapsto \mathbf{P}(t)$ and any probability distribution $\pi(0)$, it is possible to construct a corresponding CTMC with càdlàg paths. Even in case of a strictly substochastic transition probability function $t \mapsto \mathbf{P}(t)$, we interprete $t \mapsto \mathbf{P}(t)$ as a transition probability function for a CTMC $Y = (Y_t)_{t\geq 0}$. Similar to the corresponding construction for DTMPs, Y 'loses probability', that is, $\mathbb{P}_i(Y_n \notin E) = 1 - \sum_{j\in E} p_{ij}(t)$ may be positive.

C.3.2 The generator and the Kolmogorov differential equations

For standard transition probability functions,

$$\mathbf{Q} = (q_{ij})_{i,j\in\mathbb{N}_0} = \mathbf{P}'(0) = \lim_{h\to\infty} \frac{1}{h}(\mathbf{P}(h) - I)$$

exists with $q_{ii} \in [-\infty, 0]$, $q_{ij} \in [0, \infty)$ for $j \neq i$ and $\sum_{j\in E} q_{ij} \leq 0$ for all $i \in E$. The matrix $\mathbf{Q}$ is said to be the *generator* for $\mathbf{P}$ (or Y) and any matrix with $q_{ii} \in [-\infty, 0]$, $q_{ij} \in [0, \infty)$ for $j \neq i$ and $\sum_{j\in E} q_{ij} \leq 0$ is said to be a *generator matrix*. In case of $\sum_{j\in E} q_{ij} = 0$, $\mathbf{Q}$ is referred to as *conservative*. A state i with $q_{ii} = -\infty$ is said to be *volatile*, for $q_{ii} \in (-\infty, 0]$, i is called *stable*. Q is said to be *bounded* if $\sup_{i\in E}(-q_{ii}) < \infty$.

If all states are stable, $p'_{ij}(s)$ exists for all $i, j \in E$, $s > 0$, and the differential equations $\mathbf{P}'(t + s) = \mathbf{P}(t)\mathbf{P}'(s)$ and $\mathbf{P}'(s + t) = \mathbf{P}'(s)\mathbf{P}(t)$ hold for $t \geq 0$ and $s > 0$. With Fatou's Lemma, we obtain the *Kolmogorov forward inequalities* $\mathbf{P}'(t) \geq \mathbf{P}(t)\mathbf{Q}$ and the *Kolmogorov backward inequalities* $\mathbf{P}'(t) \geq \mathbf{Q}\mathbf{P}(t)$. The forward inequalities directly imply $\pi'(t) \geq \pi(t)\mathbf{Q}$. In case of a finite state space E or a bounded generator, we have equality, that is $\pi'(t) = \pi(t)\mathbf{Q}$, $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$ (Kolmogorov forward equations) and $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$ (Kolmogorov backward equation), and we have $\mathbf{P}(t) = e^{t\mathbf{Q}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{Q}^n$.

C.3.3 Feller process and regularity

If E is not finite and if Q is not bounded, but conservative, the Kolmogorov backward equations still hold, but we cannot define $e^{t\mathbf{Q}}$ anymore. Hence, two question arise:

- Does Q define a transition probability function $t \mapsto \mathbf{P}(t)$ with $\mathbf{P}'(0) = \mathbf{Q}$? Such a transition probability function will be referred to as $\mathbf{Q}$-*process*.
- If such a function exist, is it unique?

Let $\mathbf{Q}$ be a generator matrix, and let all states be stable. Using $p_{ij}(0) = \delta_{ij}$, the Kolmogorov forward and backward equations can be rewritten as

$$\begin{aligned} p_{ij}(t) &= \delta_{ij}e^{q_{jj}t} + \sum_{k\in E\setminus\{j\}} \int_0^t p_{ik}(s)q_{kj}e^{q_{jj}(t-s)}\,ds \qquad i, j \in E,\ t \geq 0 \qquad \text{and} \\ p_{ij}(t) &= \delta_{ij}e^{q_{ii}t} + \sum_{k\in E\setminus\{i\}} \int_0^t e^{q_{ii}(t-s)}q_{ik}p_{kj}(s)\,ds, \qquad i, j \in E,\ t \geq 0. \end{aligned}$$

Now we can use either of these integral equations for a fixpoint recursion (both are equivalent, this is not trivial), that is we put $f_{ij}^{(0)}(t) = 0$ and

$$f_{ij}^{(n+1)}(t) = \delta_{ij}e^{q_{jj}t} + \sum_{k\in E\setminus\{j\}} \int_0^t f_{ik}^{(n)}(s)q_{kj}e^{q_{jj}(t-s)}\,ds = \delta_{ij}e^{q_{ii}t} + \sum_{k\in E\setminus\{i\}} \int_0^t e^{q_{ii}(t-s)}q_{ik}f_{kj}^{(n)}(s)\,ds.$$

Finally, we set $f_{ij}(t) = \lim_{n\to\infty} f_{ij}^{(n)}(t)$ and $F(t) = (f_{ij}(t))$. Then $F(t)$ is well-defined ($f_{ij}^{(n)}(t)$ is bounded and monotone in n), solves both Kolmogorov forward and backward equations (monotone convergence) and is a transition function (the proof of Chapman-Kolmogorov requires some calculation) with $F'(0) = \mathbf{Q}$. Furthermore, any other solution $Z(t)$ of the Kolmogorov forward inequalities (or the Kolmogorov backward inequalities) with $Z(0) = I$ meets $z_{ij}(t) \geq f_{ij}(t)$ for all $i, j \in E$ and $t \geq 0$.

$F(t)$ is referred to as *Feller process*. The above statements say that $F(t)$ is well-defined and a $\mathbf{Q}$-process, and hence, $\mathbf{Q}$-processes exist in any case (as long as all states are stable). Furthermore, F is the minimal $\mathbf{Q}$-process. Hence, if F is stochastic, that is, if all matrices $F(t)$ are stochastic, there is no other $\mathbf{Q}$-process. Thus, it makes sense that $\mathbf{Q}$ is called *regular* (or *non-explosive*) if the corresponding Feller process F is stochastic.

The Feller process is useful for theoretical considerations. For deciding whether or not a matrix $\mathbf{Q}$ is regular, the easiest way is applying *Reuter's criterion* for regularity: A conservative generator matrix $\mathbf{Q}$ is regular if and only if the system $\mathbf{Q}x = \lambda x$ has no non-trivial nonnegative bounded solution for some $\lambda > 0$.

C.3.4 The embedded jump chain

A CTMC Y with regular generator $\mathbf{Q}$ is uniquely characterized by its *embedded jump chain* and the sequence of times between jumps.

- If $Y_0 = i$ and $q_{ii} \in (-\infty, 0)$, the first jump occurs at a time $V_i \sim \text{Exp}(-q_{ii})$.
- If $Y_0 = i$ and $q_{ii} = 0$, the CTMC will stay in i forever.
- If $Y_0 = i$ and $q_{ii} \in (-\infty, 0)$, the CTMC will leave state i at time V_i with $\mathbb{P}(Y_{V_i} = j) = \frac{q_{ij}}{-q_{ii}}$.
- Hence, we define the embedded jump chain by its transition probability matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$ with
$$p_{ij} = \begin{cases} \frac{q_{ij}}{-q_{ii}} + \delta_{ij}, & q_{ii} \in (-\infty, 0), \\ \delta_{ij}, & q_{ii} = 0. \end{cases}$$
- Let Λ be a diagonal matrix with entries $\Lambda_{xx} = \frac{1}{-q_{xx}}$ for $q_{xx} \neq 0$, and $\Lambda_{xx} = 1$ for $q_{xx} = 0$. Then we have the relationships $\mathbf{P} = \Lambda\mathbf{Q} + I$ and $\mathbf{Q} = \Lambda^{-1}(\mathbf{P} - I)$.

$\mathbf{P}$ is stochastic if and only if $\mathbf{Q}$ is conservative. Many considerations concerning DTMCs extend to CTMCs by considering the embedded jump chain.

C.3.5 Hitting probabilities and first hitting times

Again, we define the first hitting time $\sigma_C = \inf\{t \geq 0 : \ Y_t \in C\}$ for $C \subset E$. Then the probability of $\sigma_C < \infty$ is the same as for the embedded jump chain, and the inequality $h(x) \geq (\mathbf{P}h)(x)$ for $x \in E \setminus C$ yields

$$(\mathbf{Q}h)(x) = \Lambda^{-1}((\mathbf{P}h)(x) - h(x)) \leq \Lambda^{-1}(h(x) - h(x)) \leq 0.$$

The same is true for equality. Hence, we obtain

Theorem C.3.1. *Let Y be a CTMC with stable and regular generator matrix $\mathbf{Q}$, and let $C \subset E$.*

- *For $g : E \to \mathbb{R}$ with $x \mapsto g(x) = \mathbb{P}_x(\sigma_C < \infty)$, we have*

$$\sum_{y \in E} q_{xy} g(y) = 0, \qquad x \in E \setminus C,$$

and for any other bounded solution $h : E \to \mathbb{R}_{\geq 0}$ of

$$\begin{aligned} h(x) &\geq 1, \qquad x \in C, \\ 0 &\geq \sum_{y \in E} q_{xy} h(y), \end{aligned}$$

we have $h(x) \geq g(x)$ for all $x \in E$.

- *Define $G : E \times C \to \mathbb{R}$ with $(x, y) \mapsto G_{xy} = \mathbb{P}_x(\sigma_C < \infty, Y_{\sigma_C} = y)$. For $x \in C$, we have $G_{xy} = \delta_{xy}$, and for $x \notin C$, we have*

$$\sum_{z \in E} q_{xz} G_{zy} = 0.$$

For any other function $H : E \times C \to \mathbb{R}$ with

$$\begin{aligned} H_{xy} &\geq \delta_{xy}, \qquad x \in C, \\ 0 &\geq \sum_{z \in E} q_{xz} H_{zy}, \qquad x \in E \setminus C, \end{aligned}$$

we have $H_{xy} \geq G_{xy}$.

The result for mean hitting times can be extended to CTMCs, too.

Theorem C.3.2. *Let Y be a CTMC with a stable and regular generator matrix $\mathbf{Q}$, let $C \subset E$, and let $f : E \to [0, \infty)$. If there is a function $h : E \setminus C \to [0, \infty]$ with*

$$0 \geq \sum_{y \in E \setminus C} q_{xy} h(y) + f(x), \qquad x \in E \setminus C,$$

we have $g(x) := \mathbb{E}_x \left[\int_0^{\sigma_C} f(Y_t)\, dt \right] \leq h(x)$ and

$$0 = \sum_{y \in E \setminus C} q_{xy} g(y) + f(x)$$

for all $x \in E \setminus C$. In particular, if $h(x) < \infty$ for some x, we have $g(x) < \infty$.

Proof. Let V be the time until the first jump occurs. Then

$$\begin{aligned} g(x) &= \mathbb{E}_x\left[\int_0^{\sigma_C} f(Y_t)\,dt\right] \\ &= \mathbb{E}_x\left[\int_0^{V} f(Y_t)\,dt\right] + \sum_{y\in E\setminus\{x\}} p_{xy}\mathbb{E}_y\left[\int_0^{\sigma_C} f(Y_t)\right] \\ &= \mathbb{E}_x\left[V\right]\cdot f(x) + \sum_{y\in E\setminus\{x\}} p_{xy}g(y). \end{aligned}$$

Let $q_{xx} \neq 0$. Then, conditioned on $Y_0 = x$, we have $V \sim \mathrm{Exp}(-q_{xx})$, and hence

$$g(x) = \frac{f(x)}{-q_{xx}} + \sum_{y\in E\setminus\{x\}} \frac{q_{xy}}{-q_{xx}} g(y),$$

yielding

$$0 = f(x) + \sum_{y\in E} q_{xy}g(y).$$

For $q_{xx} = 0$, we have $0 \geq \sum 0h(y) + f(x) = f(x)$, that is $f(x) = 0$. Then trivially $\sum q_{xy}g(y) + f(x) = 0$.

For proving the minimality of g, we use the transition probability matrix $\mathbf{P}$ of the embedded jump chain and two facts:

- For $q_{xx} < 0$, it is easy to prove that

$$\sum_{y\in E} q_{xy}h(y) + f(x) \leq 0 \qquad \Leftrightarrow \qquad \sum_{y\in E} p_{xy}h(y) + \frac{f(x)}{-q_{xx}} \leq h(x).$$

  For $q_{xx} = 0$ and $f(x) = 0$, this statement remains true, where we define $\frac{0}{0} = 0$. As pointed out above, $q_{xx} = 0$ and $f(x) > 0$ will not occur under the conditions of the Theorem.

- Let $\sigma_C^* = \inf\{n \in \mathbb{N}_0 : X_n \in C\}$ with the embedded jump chain X, and let $f^*(x) = \frac{f(x)}{-q_{xx}}$ (again, let $\frac{0}{0} = 0$). Then we have

$$g(x) = \mathbb{E}_x\left[\sum_{k=0}^{\sigma_C^*-1} f^*(X_k)\right],$$

Since g is minimal for the embedded jump chain with cost function f^*, the equivalence in the first point yields that g is minimal for the CTMC with cost function f. □

Note that we have used the same variables g and G as for DTMPs. This is due to the fact that we have the same interpretation in both cases (hitting probabilities and mean costs (measured by f) before the first hitting on C occurs).

C.3.6 Irreducibility and classification

In the literature, irreducibility of a CTMC with generator matrix $\mathbf{Q}$ is often characterized by the condition, that for any $\mathbf{Q}$-process $\mathbf{P}$ and all $i, j \in E$, there is some $t > 0$ with $p_{ij}(t) > 0$. This characterization is equivalent to the existence of $i_0 = i, i_1, \ldots, i_n = j$ with $q_{i_{r-1},i_r} > 0$ for all $r = 1, \ldots, n$. Finally, this condition is equivalent to irreducibility of the embedded jump chain. So even if $\mathbf{Q}$ is not regular, irreducibility can be written in terms of the embedded jump chain. Hence, the following definition makes sense:

Theorem C.3.3. *Let Y be a CTMC with generator matrix $\mathbf{Q}$ and let $\mathbf{P}$ be the transition probability matrix for the embedded jump chain. Then X is said to be φ-irreducible if $\mathbf{P}$ is φ-irreducible.*

At some points, we will identify further communication properties of the CTMC and its embedded jump chain. A term that depends directly on communication properties is that of absorption: A set $C \subset E$ is said to be *absorbing* if it is absorbing for the embedded jump chain, which is equivalent to $\sum_{y \in C} q_{xy} = 0$ for all $x \in C$, that is the restriction of $\mathbf{Q}$ to $C \times C$ is a conservative generator. In particular, an absorbing state x is characterized by $q_{xx} = 0$. If x is absorbing, $p_{xx}(t) = 1$ for all $t \geq 0$. As for DTMPs, a φ-irreducible CTMC cannot contain two distinct absorbing sets.

For (positive) recurrence and transience, there are different possibilities for the definition (as there are for DTMCs).

- Let $\eta_x = \int_0^\infty \mathbf{1}_{\{x\}}(Y_t)\, dt$ be the time the CTMC spends in state x. We say that x is *recurrent* if $\mathbb{E}_x[\eta_x] = \infty$, otherwise we call x *transient*.
- Let τ_x be the first return time to state x, that is,

  $$\tau_x = \inf\{t > 0 : \ Y_t = x, \ \exists s \in [0, t) : \ Y_s \neq x\}.$$

  Then $L(x, x) := \mathbb{P}_x(\tau_x < \infty) = 1$ for non-absorbing recurrent x and $L(x, x) < 1$ for transient x.
- Again, if $x \leftrightarrow y$, either both states are recurrent or both are transient.
- If Y is irreducible and all states are transient, Y is said to be transient. If all states are recurrent, X is called recurrent.
- x is recurrent for the CTMC if and only if x is recurrent for the embedded jump chain. Hence, the easiest way to define the terms is as follows: A set $B \subset E$ is said to be *(Harris) recurrent* or *(uniformly) transient* if and only if it is (Harris) recurrent or (uniformly) transient for the embedded jump chain.
- Furthermore, we define a φ-irreducible CTMC to be (Harris) recurrent or transient if the embedded jump chain is. For $\varphi = \zeta$, this definition coincides with the usual one.
- Positive recurrence cannot be defined by using the embedded jump chain. Quite popular is defining $x \in E$ as *positive recurrent* if $\lim_{t \to \infty} p_{xx}(t) > 0$. Any absorbing state is positive

recurrent; for non-absorbing states, the definition is equivalent to $\mathbb{E}_x[\tau_x] < \infty$. Again, $x \leftrightarrow y$ implies that either both x and y are positive recurrent or both are not. In particular, an irreducible CTMC is defined as positive recurrent if all states are positive recurrent. In this case, an invariant distribution (see below) exists and coincides with the limit distribution $\pi(\infty)$.

- Hence, it makes sense to give a slightly more general definition: A φ-irreducible CTMC is said to be *positive recurrent* if there is an invariant distribution π. In this case (see below), we have $\lim_{t\to\infty} p_{xx}(t) > 0$ if and only if $\{x\} \in \mathcal{E}^+$.

C.3.7 Invariant and subinvariant measures

Suppose that $\psi^* = (\psi^*_x)_{x\in E}$ is an (sub)invariant measure for the embedded jump chain, that is

$$\sum_{x\in E} \psi^*_x p_{xy} \leq \psi^*_y, \qquad x \in E.$$

Define $\psi_x = \frac{\psi^*_x}{-q_{xx}}$ for $q_{xx} \neq 0$ and $\psi_x \in (0,\infty)$ arbitrary for $q_{xx} = 0$. Then we have $\psi\mathbf{Q} \overset{(\leq)}{=} 0$:

- If y is non-absorbing, we have

$$\sum_{x\in E} \psi_x q_{xy} = \sum_{x\in E} \psi^*_x \frac{q_{xy}}{-q_{xx}} = \sum_{x\in E\setminus\{y\}} \psi^*_x p_{xy} - \psi^*_y \overset{(\leq)}{=} 0.$$

  This derivation is right even if there is some absorbing $x \neq y$ since then $q_{xy} = 0$.

- If y is absorbing, the subinvariance yields $\psi^*_x = 0$ for all $x \neq y$ with $p_{xy} = 0$, that is, $\psi_x = 0$ for all $x \neq y$ with $q_{xy} > 0$. Hence, we directly obtain $\sum_{x\in E} \psi_x q_{xy} = 0$.

On the other hand, assume that $\sum_{x\in E} \psi_x q_{xy} \overset{(\leq)}{=} 0$ for all $y \in E$. With $\psi^*_x = \psi_x \cdot (-q_{xx})$, we obtain

$$\sum_{x\in E} \psi^*_x p_{xy} = \sum_{x\in E\setminus\{y\}} \psi_x \cdot (-q_{xx}) \frac{q_{xy}}{-q_{xx}} = \sum_{x\in E\setminus\{y\}} \psi_x q_{xy} \overset{(\leq)}{=} \psi_y \cdot (-q_{yy}) = \psi^*_y.$$

For CTMCs, a non-trivial measure ψ is said to be *subinvariant* if $\psi\mathbf{Q} \leq 0$, and in case of $=$, it is said to be *invariant*. Again, an invariant probability measure π is said to be an invariant distribution. Due to the one-to-one-relation between (sub)invariant measures for CTMCs and the embedded jump chain, many results can be extended easily to CTMCs, e.g. in case of φ-irreducibility and recurrence, all subinvariant measures are invariant and unique up to a constant multiple. With respect to positive recurrence, it is important to state that positive recurrence of the CTMC and its embedded jump chain do not imply each other, we have $\psi(E) < \infty \not\Rightarrow \psi^*(E) < \infty \not\Rightarrow \psi(E) < \infty$.

With considerations as above, we can extend Theorem C.2.4 to CTMCs.

Theorem C.3.4. *Let Y be a CTMC with regular generator $\mathbf{Q}$, let $C \subset E$, let $\mathbb{P}(\sigma_C < \infty | X_0 = x) > 0$ for all $x \in E$, and define*

$$F(x,y) = \begin{cases} -q_{xx} \cdot \mathbb{E}_x \left[\int_0^{\tau_C} \mathbf{1}_{\{y\}}(Y_t)\, dt \right] = -q_{xx} \int_0^\infty \mathbb{P}_x(Y_t = y, \tau_C > t), & x \text{ non-absorbing}, \\ \delta_{xy}, & x \text{ absorbing} \end{cases}$$

for $x \in C, y \in E$, where τ_C is the time for which the first jump (from a state $\in C$ or $\notin C$) into a state $\in C$ occurs. Furthermore, let

$$K(x,y) = \sum_{z \in E} F(x,z) q_{zy}, \qquad x, y \in C.$$

a) $F : C \times E \to [0,\infty)$ meets

$$\begin{aligned} F(x,y) &= \delta_{xy}, \qquad y \in C, \\ 0 &= \sum_{z \in E} F(x,z) q_{zy}, \qquad y \in E \setminus C. \end{aligned}$$

b) Let $H : C \times E \to [0,\infty]$ meet

$$\begin{aligned} H(x,y) &\geq \delta_{xy}, \qquad y \in C, \\ 0 &\geq \sum_{z \in E} H(x,z) q_{zy}, \qquad y \in E \setminus C. \end{aligned}$$

Then $H(x,y) \geq F(x,y)$ for all $x \in C, y \in E$.

c) $K : C \times C \to \mathbb{R}$ is a generator which is conservative if and only if C is Harris recurrent.

d) Let ψ_C be a (sub)invariant measure for K. Then $\psi := \psi_C F$ is a (sub)invariant measure for $\mathbf{Q}$.

e) If Y is φ-irreducible, K is $\varphi|_C$-irreducible.

Proof. Let X be the embedded jump chain, let $\sigma_C^* = \inf\{n \in \mathbb{N} : \ X_n \in C\}$, and let

$$F^*(x,B) = \mathbb{E}_x \left[\sum_{n=0}^{\sigma_C^* - 1} \mathbf{1}_B(X_n) \right] = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n \in B, \tau_C > n),$$

that is, F^* is defined as F was in Theorem C.2.4. Furthermore, let $T = F^* \mathbf{P}$ (see Theorem C.2.4 again).

a) Due to $\mathbb{P}_y(\sigma_C < \infty) > 0$ for $y \in E \setminus C$, no state $y \in E \setminus C$ can be absorbing. Since the time a CTMC spends in state x is $\mathrm{Exp}(-q_{xx})$-distributed, $F(x,y) = \delta_{xy}$ for $y \in C$ is obvious. Furthermore, this fact yields $F(x,y) = -q_{xx} F^*(x,y) \cdot \frac{1}{-q_{yy}}$, and hence, for $y \notin C$, we obtain

$$\sum_{z \in E} F(x,z) q_{zy} = -q_{xx} \sum_{z \in E} F^*(x,z) \left(p_{zy} - \delta_{zy} \right) = -q_{xx} \left(F^*(x,y) - F^*(x,y) \right) = 0.$$

b) If x is absorbing, put $H^*(x,y) = H(x,y)$. Otherwise, define $H^*(x,y) = \frac{1}{-q_{xx}} H(x,y)(-q_{yy})$. Similar as for subinvariant measures, we see that H solving the inequalities in the statement implies that H^* solves the inequalities in Theorem C.2.4, and hence, $H^*(x,y) \geq F^*(x,y)$, and $H(x,y) \geq F(x,y)$ follows directly.

c) If x is absorbing, $K(x,y) = 0$ for all $y \in C$. Otherwise, we use $F^*(x,y) = \delta_{xy}$ for $y \in C$, and obtain

$$\begin{aligned} \frac{1}{-q_{xx}} K(x,y) + \delta_{xy} &= \sum_{z \in E} \frac{1}{-q_{xx}} F(x,z) q_{zy} + \delta_{xy} \\ &= \sum_{z \in E} F^*(x,z) \frac{q_{zy}}{-q_{yy}} + \delta_{xy} \\ &= \sum_{z \in E} F^*(x,z)(p_{zy} - \delta_{zy}) + F^*(x,y) \\ &= \sum_{z \in E} F^*(x,z) p_{zy} = T(x,y). \end{aligned}$$

In any case, we have

$$K(x,y) = (-q_{xx})(T(x,y) - \delta_{xy}).$$

Obviously, K is conservative if and only if T is stochastic, and this is equivalent to Harris recurrence of C.

d) Follows from the relationship between CTMC and the embedded jump chain.

e) Follows from the relationship between CTMC and the embedded jump chain.

□

Again, we have used the same variables F as for DTMPs. In both cases, $F(x,B)$ may be interpreted as the proportion of the time spent in B and the time spent in x before the first jump into a state of set C occurs. The matrix K does not have the same interpretation as the matrix T in the discrete time case: K is a generator, that is a matrix of rates, whereas T is a transition probability matrix.

C.3.8 Limiting behaviour and drift criteria

The statements for the limiting behaviour are very similar to the discrete-time case, things are even easier due to the fact that CTMCs cannot behave cylic, and thus, we do not have to distinguish between aperiodic and periodic behaviour.

Theorem C.3.5. *Let X be a φ-irreducible CTMC.*

- *Let X be positive Harris, and let π be the invariant distributions. Then*

$$\lim_{t \to \infty} ||\pi(t) - \pi||_{TV} = 0,$$

that is, for any bounded function $f : E \to \mathbb{R}$, we have $\lim_{t \to \infty} ||\pi(t)f - \pi f|| = 0$. In particular, $\pi(t)$ converges weakly to π, or equivalently, X_t converges in distribution to some random variable X_∞ with distribution π.

- *Let X be Harris recurrent, and let ψ denote an invariant distribution, let $f, g : E \to \mathbb{R}$ be ψ-integrable, that is $\psi |f| < \infty$ and $\psi |g| < \infty$, and let $\psi g \neq 0$. Then*

$$\lim_{t\to\infty} \frac{\int_0^t f(X_s)ds}{\int_o^t g(X_s)ds} = \frac{\psi f}{\psi g} \text{ almost surely.}$$

- *Let X be positive Harris, let π denote the invariant distribution, and let f be π-integrable. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s)ds = \pi f \text{ almost surely.}$$

- *Let X be (positive) recurrent, but not necessarily Harris. Then there is some set $N \notin \mathcal{E}^+$ such that whenever $\sum_{i\in N} \pi_i(0) = 0$, the above results hold.*

For drift criteria, we need to redefine the drift. For CTMCs, we define $d_g(x) = \mathbf{Q}g(x) = \sum_{y\in E} q_{xy}g(y)$. By means of the embedded jump chain, Tweedie [Twe75] proved drift criteria for CTMCs. He was able to show that drift conditions yield regularity, too.

Theorem C.3.6. *Let Y be a φ-irreducible CTMC with conservative generator $\mathbf{Q}$, let $g : E \to [0, \infty)$ and define its drift $d_g : E \to \mathbb{R} \cup \{\infty\}$ by*

$$d_g(x) = \mathbf{Q}g(x) = \sum_{y\in E} q_{xy}g(y),$$

let $C \in \mathcal{E}$ be finite, let $\epsilon, b > 0$, and let $f : E \to [0, \infty)$.

- *Let $\{x \in E : \ g(x) \leq r\}$ be finite for all $r > 0$, and let $d_g(x) \leq 0$ for $x \in E \setminus C$. Then $\mathbf{Q}$ is regular and Y is Harris recurrent.*

- *Let $d_g(x) \leq -\epsilon + b\mathbf{1}_C(x)$ for all $x \in E$. Then Y is positive Harris.*

Appendix D

Codes

D.1 Code for computing stationary distributions numerically

In section 6.3, invariant measures for Markov chains were computed numerically, here we present the C++-code.

```
#include <iostream>

int main(int argc, char *argv[])
{
    double rho=1.0/3;
    int n,N=100;
    double exact[N+1];
    double forward[N+1];
    double R[N+1];
    double cf[N+1];

    exact[0]=1;
    for (n=1;n<=N;n++) exact[n]=exact[n-1]*rho;

    forward[0]=1;
    forward[1]=rho;
    for (n=2;n<=N;n++) forward[n]=(rho+1)*forward[n-1]-rho*forward[n-2];

    R[N]=0;
    for (n=N-1;n>=0;n–) R[n]=rho/(rho+1-R[n+1]);
    cf[0]=1;
    for (n=1;n<=N;n++) cf[n]=cf[n-1]*R[n-1];

    for (n=0;n<=N;n++)
        printf("n=%d Exact: %.3e Forward: %.3e CF: %.3e\n",n,exact[n],forward[n],cf[n]);
```

```
    double gamma=9.0;double a=10.0;

    exact[0]=1;
    for (n=1;n<=N;n++) exact[n]=exact[n-1]*gamma/a;

    forward[0]=1;
    forward[1]=gamma/a;
    for (n=2;n<=N;n++)
        forward[n]=((a+1)*gamma*forward[n-1]-gamma*gamma*forward[n-2])/a;

    R[N]=0;
    for (n=N-1;n>=0;n–) R[n]=gamma*gamma/(gamma*(a+1)-R[n+1]*a);
    cf[0]=1;
    for (n=1;n<=N;n++) cf[n]=cf[n-1]*R[n-1];

    for (n=0;n<=N;n++)
        printf("n=%d Exact: %.3e Forward: %.3e CF: %.3e\n",n,exact[n],forward[n],cf[n]);

    system("PAUSE");
    return EXIT_SUCCESS;
}
```

Appendix E

Symbols and notations

- We define 1 to be the smallest natural number, that is, $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- By $\mathbb{N}_0$, we refer to
$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$
- We use the notation $\mathbb{N}_{n,N} := \{n, \ldots, N\}$ for $n, N \in \mathbb{N}_0$, $n \leq N$, and $\mathbb{N}_{n,N} = \emptyset$ for $n > N$.
- We use $\prod_{i=a}^{b} r_i$ for the 'forward-product', that is,
$$\prod_{i=a}^{b} r_i = r_a r_{a+1} \ldots r_b.$$
- For the 'backward-product', we write
$$\prod_{a}^{i=b} r_i = r_b r_{b-1} \ldots r_a.$$
- $\mathbf{1}$ denotes the function with constant value 1 or the vector with value 1 in each component.
- $\mathbf{1}_A$ denotes the indicator function of set A, that is, $\mathbf{1}_A(x) = 1$ for $x \in A$ and $\mathbf{1}_A(x) = 0$ for $x \notin A$.
- If not indicated otherwise, probability measures are denoted by $\mathbb{P}$ or $\mathbb{P}_x$. The corresponding expectations are denoted by $\mathbb{E}[f] = \int f d\mathbb{P}$ and $\mathbb{E}_x f = \int f d\mathbb{P}_x$.

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